Modern Physics

Week 3 - Tutorial Solutions

14^{th} September, 2012

Problem 1. Show that $E^2 = (mc^2)^2 + (pc)^2$
Solution: Proof done in class.

Problem 2. A particle of mass M decays at rest into two particles, each with mass m moving in opposite directions with velocity v. Find speed of each of the particle in rest frame.
Solution: We know that energy must be conserved in all interactions. Therefore, writing the equation for energy conversation:

$$\gamma_0 Mc^2 = \gamma_v mc^2 + \gamma_v mc^2$$

Mass M is at rest, therefore we will put $\gamma_0 = 1$.

$$Mc^2 = 2\gamma_v mc^2$$

$$M = 2\gamma_v m$$

$$M = 2\frac{m}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\sqrt{1 - \frac{v^2}{c^2}} = \frac{2m}{M}$$

$$1 - \frac{v^2}{c^2} = \left(\frac{4m^2}{M^2}\right)$$

$$v = c\sqrt{1 - \left(\frac{2m}{M}\right)^2}$$

Problem 3. Mass of a deuteron is measured to be 1875.6 MeV/c^2. The sum of masses of the constituents of deuteron (a proton and a neutron) is 1877.9 MeV/c^2. Observing that the sum of masses of constituent particles is more than the mass of deuteron, argue why the existence of deuteron is still possible. Also tell if deuteron will disintegrate into its constituent particles on its own, and if not, then why?
Solution: In the process of making a deuteron from a neutron and a proton, energy is released which is exactly equal to the energy difference of $\delta E = 1877.9$ MeV - 1875.6 MeV = 2.3 MeV. Once a deuteron is made its rest energy is less than the sum of rest energies of a neutron and a proton. Thus a deuteron is a stable particle and does not by itself disintegrate into two.
This effect can also be seen by investigating the formula derived in Problem 2. If $2m > M$ v becomes complex which makes no sense in our physical world. Thus for a particle to disintegrate into two it is necessary that the rest mass of the bigger particle must be greater than the sum of the two constituent particles (as it was in the case discussed in problem 2.)
Note: Consider the case when M is exactly equal to 2m. v is zero in this case and thus the two particles will have no kinetic energy. They will be in the same position as the initial particle unless some energy is provided from the outside and the new particles are made to move. .

Problem 4.

(*a*) What is the kinetic energy acquired by a particle of charge q starting from rest in a uniform electric field when it falls through an electrostatic potential difference of V_0 volts? The work done on the charge q by the electric field $\mathbf{E}$ in a displacement $d\mathbf{l}$ is

$$dW = q\mathbf{E} \cdot d\mathbf{l}.$$

Let the uniform field be in the x-direction so that $\mathbf{E} \cdot d\mathbf{l} = E_x\, dx$ and

$$W = \int qE_x\, dx.$$

Now $E_x = -(dV/dx)$, where V is the electrostatic potential, so that

$$\begin{aligned} W &= -\int q\frac{dV}{dx}dx = -q\int dV = -q(V_f - V_i) \\ &= q(V_i - V_f) = qV_0 \end{aligned}$$

where V_0 is the difference between the initial potential V_i and the final potential V_f. The kinetic energy acquired by the charge is equal to the work done on it by the field so that

$$K = qV_0. \tag{3-23}$$

Notice that we have implicitly assumed that the charge q of the particle is a constant, independent of the particle's motion.

(*b*) Assume the particle to be an electron and the potential difference to be 10^4 volts. Find the kinetic energy of the electron, its speed, and its mass at the end of the acceleration.

The charge on the electron is $e = -1.602 \times 10^{-19}$ coulomb. The potential difference is now a rise, $V_i - V_f = -10^4$ volts, a negative charge accelerating in a direction opposite to $\mathbf{E}$. Hence, the kinetic energy acquired is

$$K = qV_0 = (-1.602 \times 10^{-19})(-10^4) \text{ joules} = 1.602 \times 10^{-15} \text{ joules}.$$

From Eq. 3-16, $K = mc^2 - m_0c^2$, we obtain

$$\frac{K}{c^2} = (m - m_0)$$

or

$$(1.602 \times 10^{-15} \text{ joules}/8.99 \times 10^{16}\, m^2/\text{sec}^2) = m - m_0 = 1.78 \times 10^{-32} \text{ kg}$$

and, with $m_0 = 9.109 \times 10^{-31}$ kg, we find the mass of the moving electron to be

$$m = (9.109 + 0.178) \times 10^{-31} \text{ kg} = 9.287 \times 10^{-31} \text{ kg}.$$

Notice that $m/m_0 = 1.02$, so that the mass increase due to the motion is about 2 percent of the rest mass.

From Eq. 3-10, $m = m_0/\sqrt{1 - u^2/c^2}$, we have

$$\frac{u^2}{c^2} = \left[1 - \left(\frac{m_0}{m}\right)^2\right] = \left[1 - \left(\frac{9.109}{9.287}\right)^2\right] = 0.038$$

or

$$u = 0.195c = 5.85 \times 10^7 \text{ m/sec.}$$

The electron acquires a speed of about one-fifth the speed of light.

Problem 5.

(*a*) Show that, in a region in which there is a uniform magnetic field, a charged particle entering at right angles to the field moves in a circle whose radius is proportional to the particle's momentum.

Call the charge of the particle q and its rest mass m_0. Let its velocity be $\mathbf{u}$. The force on the particle is then

$$\mathbf{F} = q\mathbf{u} \times \mathbf{B}$$

which is at right angles both to $\mathbf{u}$ and to $\mathbf{B}$, the magnetic field. Hence, from Eq. 3-22, the acceleration,

$$\mathbf{a} = \frac{\mathbf{F}}{m} = \frac{q}{m}\mathbf{u} \times \mathbf{B},$$

is in the same direction as the force. Because the acceleration is always at right angles to the particle's velocity $\mathbf{u}$, the speed of the particle is constant and the particle moves in a circle. Let the radius of the circle be r, so that the centripetal acceleration is u^2/r. We equate this to the acceleration obtained from above, $a = quB/m$, and find

$$\frac{quB}{m} = \frac{u^2}{r}$$

or

$$r = \frac{mu}{qB} = \frac{p}{qB}. \tag{3-24}$$

Hence, the radius is proportional to the momentum $p(= mu)$.

Notice that both the equation for the acceleration and the equation for the radius (Eq. 3-24) are identical in form to the classical results, but that the rest mass m_0 of the classical formula is replaced by the relativistic mass $m = m_0/\sqrt{1 - u^2/c^2}$.

How would the motion change if the initial velocity of the charged particle had a component parallel to the magnetic field?

(*b*) Compute the radius, both classically and relativistically, of a 10 Mev electron moving at right angles to a uniform magnetic field of strength 2.0 webers/m^2.

Classically, we have $r = m_0u/qB$. The classical relation between kinetic energy and momentum is $p = \sqrt{2m_0K}$ so that

$$p = \sqrt{2m_0K} = \sqrt{2 \times 9.1 \times 10^{-31}\ \text{kg} \times 10\ \text{Mev} \times 1.6 \times 10^{-13}\ \text{joule/Mev}} = 17 \times 10^{-22}\ \text{kg m/sec.}$$

Then

$$r = \frac{m_0u}{qB} = \frac{p}{qB} = \frac{17 \times 10^{-22}}{1.6 \times 10^{-19} \times 2.0}\ \text{meter} = 5.3 \times 10^{-3}\ \text{meter} = 0.53\ \text{cm.}$$

Relativistically, we have $r = mu/qB$. The relativistic relation between kinetic energy and momentum (Eq. 3-18*a*) may be written as

$$p = \frac{1}{c}\sqrt{(K + m_0c^2)^2 - (m_0c^2)^2}.$$

Here, the rest energy of an electron, m_0c^2, equals 0.51 Mev, so that

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$$p = \frac{1}{3 \times 10^8}\sqrt{(10 + 0.51)^2 - (0.51)^2}\,\frac{\text{Mev-sec}}{\text{meter}}\ (1.16 \times 10^{-13}\ \text{joule/Mev}) = 5.6 \times 10^{-21}\ \text{kg} - \text{m/sec.}$$

Then

$$r = \frac{mu}{qB} = \frac{p}{qB} = \frac{5.6 \times 10^{-21}}{1.6 \times 10^{-19} \times 2.0}\ \text{meter} = 1.8 \times 10^{-2}\ \text{meter} = 1.8\ \text{cm.}$$

Experiment bears out the relativistic result .