Modern Physics

Week 2 - Tutorial Solutions

7^{th} September, 2012

Problem 1. Consider a railway man standing at the middle of a freight car of length 2L. He flicks on his lantern and a light pulse travels out in all directions with the velocity c. Find the time of arrival of the light pulse at each end of the freight car in the following two frames: i. Frame of the railway man standing at the middle of the freight car. ii. Frame of a man standing outside the freight car, whose frame is coincident with railway man's frame initially, before the train starts to move with speed v in his frame.

Solution: Let the frame of railway man be S, and the frame of a man standing outside the freight car, with respect to whom freight car is moving with speed v, be S'. Then we find the coordinates of the event in frame S first, and then we will see what are the coordinates of those same events in frame S'.

In frame S: $\Delta x_1 = -L$, $\Delta x_2 = L$. These correspond to the distance of the end points of the freight car from the origin of frame S, which we take to be at the middle of the freight car. We find amount of time light took to reach either end in the frame of S:

$$\Delta t_1 = \frac{L}{c} = T \quad , \quad \Delta t_2 = \frac{L}{c} = T$$

Now, to find the amount of time measured in frame of S', between these two events - light being shot from the middle of the freight car, and it being received at the distant end of the car; we use Lorentz transformations:

$$\Delta t_1' = \gamma \left(\Delta t_1 - \frac{v \Delta x_1}{c^2} \right)$$

$$\Delta t_1' = \gamma \left(T + \frac{vL}{c^2} \right)$$

$$\Delta t_1' = \frac{1}{\sqrt{1 - v^2/c^2}} \left(T + \frac{v}{c} T \right)$$

$$\Delta t_1' = T \sqrt{\frac{1 + v/c}{1 - v/c}}$$

Similarly, we use Lorentz transformation to find the time taken (in frame S') by light to reach the near end of the freight car from the middle of the car, as:

$$\Delta t_2' = \gamma \left(\Delta t_2 - \frac{v \Delta x_2}{c^2} \right)$$

$$\Delta t_2' = \gamma \left(T - \frac{vL}{c^2} \right)$$

$$\Delta t_2' = \gamma \left(T - \frac{vT}{c} \right)$$

$$\Delta t_2' = \frac{T}{\sqrt{1 - \frac{v^2}{c^2}}} \left(1 - \frac{v}{c} \right)$$

$$\Delta t_2' = T \sqrt{\frac{1 - v/c}{1 + v/c}}$$

Thus we see that $\Delta t_2' < \Delta t_1'$. Hence light reached the back end of the train earlier than it reached the front end, in the frame of S'.

Problem 2. For any two events A and B, can we always go to frame where they are simultaneous?

Solution:

Two events A and B have the following coordinates in the x, y system.

Event A:

x_A, t_A.

Event B:

x_B, t_B.

(For both events, $y = 0$.)

The distance L and time T separating the events in the x, y system are

$$L = x_B - x_A$$
$$T = t_B - t_A.$$

For concreteness, we take L and T to be positive. To find the coordinates in the x', y' system we use the Lorentz transformations, Eq. (12.1):

$$x_A' = \gamma(x_A - vt_A)$$
$$t_A' = \gamma\left(t_A - \frac{vx_A}{c^2}\right)$$
$$x_B' = \gamma(x_B - vt_B)$$
$$t_B' = \gamma\left(t_B - \frac{vx_B}{c^2}\right).$$

The distance L' between the events in the x', y' system is

$$L' = x_B' - x_A'$$
$$= \gamma[x_B - x_A - v(t_B - t_A)]$$
$$L' = \gamma(L - vT).$$

Similarly,

$$T' = \gamma\left(T - \frac{vL}{c^2}\right).$$

Now, if we want to go to a frame where the two events A and B are simultaneous we simply want $t_A' = t_B'$. For that we want T' equal to zero (since T' was initially defined as $t_B' - t_A'$).
Now it's a matter of observation that if we put $v = \frac{c^2 T}{L}$ in the equation above, we get T' = 0. i.e. for any two events (x_A, t_A) and (x_B, t_B) we can go to frame that is moving with speed $v = \frac{c^2(t_B - t_A)}{(x_B - x_A)}$ with respect to the original frame, and in which the two events will indeed be simultaneous.

Problem 3. The solution to this problem has already been uploaded. It can be found in Mudassir's tutorials (Tutorial no. 2, Problem no. 1).

Problem 4. Solution to this problem will be separately uploaded.

SOLUTION

a. From the statement of the problem,

$$x_o = x_o' = 0 \qquad x_A = 0 \qquad x_B = 2 \times 10^6 \text{ light-years}$$
$$t_o = t_o' = 0 \qquad t_A = 2000 \text{ years} \qquad t_B = 2000 \text{ years}$$

We want the speed v_{rel} of the Enterprise such that $t_B' = 0$. The first two Lorentz transformation equations (L-10) with $t_B' = 0$ become

$$t_B = v_{rel}\gamma x_B'$$
$$x_B = \gamma x_B'$$

We do not yet know the value of x_B'. Solve for v_{rel} by dividing the two sides of the first equation by the respective sides of the second equation. The unknown x_B' drops out (along with γ), and we are left with v_{rel} in terms of the known quantities t_B and x_B:

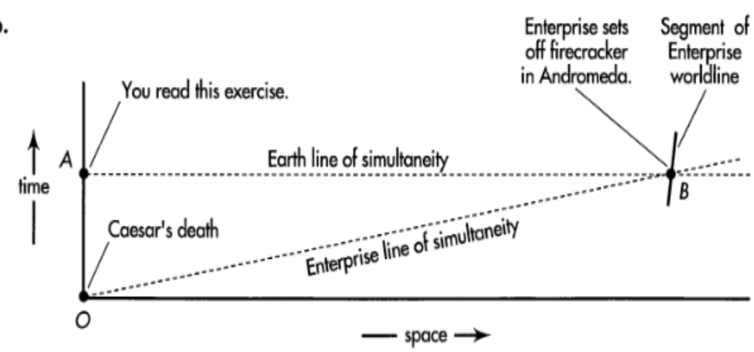
$$v_{rel} = \frac{t_B}{x_B} = \frac{2 \times 10^3 \text{ years}}{2 \times 10^6 \text{ years}} = 10^{-3} = 0.001$$

This is the desired speed v_{rel} between Earth and Enterprise frames. This velocity is a positive quantity, so the Enterprise moves in the positive x-direction, namely away from Earth.

Surprised to see a speed given as the ratio of a time separation to a space separation: t_B/x_B? Then realize that x_B and t_B are not displacements of any particle. Nothing can travel the distance x_B in the time t_B, as discussed in **d**. The goal here is to find a frame in which Caesar's death and the firecracker explosion are simultaneous. For this limited purpose the rocket speed $v_{rel} = t_B/x_B$ is correct.

Why is the relative velocity v_{rel} so small compared with the speed of light? Because of the large denominator x_B in the equation that leads to this value. Consider the string of Earth clocks stretching toward Andromeda when all Earth clocks read zero time (Caesar's death). Enterprise clocks read (from equations L-11 with $t = 0$) as follows: $t' = -v_{rel}\gamma x$. This is an example of the relativity of simultaneity (Section 3-4). The farther the x-distance from Earth, the earlier will Enterprise clock read. With $x = 2$ million light-years, the relative speed v_{rel} does not have to be large to carry Enterprise time back 2000 years for Earth.

b.



Earth spacetime diagram, showing events O, A, and B. Not to scale.

c. We need the value of gamma, γ, for the inverse Lorentz transformation equation (L-11). This value is very close to unity, and from it come t_B' and x_B'.

$$\gamma = \frac{1}{[1 - v_{\text{rel}}^2]^{1/2}} = \frac{1}{[1 - (10^{-3})^2]^{1/2}} = \frac{1}{[1 - 10^{-6}]^{1/2}} \approx 1 + \frac{10^{-6}}{2}$$

$$t_B' = -v_{\text{rel}}\gamma x_B + \gamma t_B = \gamma(-10^{-3} \times 2 \times 10^6 + 2 \times 10^3)$$

$$= \gamma(-2 \times 10^3 + 2 \times 10^3) = 0 \text{ years}$$

$$x_B' = \gamma x_B - v_{\text{rel}}\gamma t_B = \gamma(2 \times 10^6 - 10^{-3} \times 2 \times 10^3) = 2\gamma(1 - 10^{-6})\, 10^6$$

$$= 2\left(1 + \frac{10^{-6}}{2}\right)(1 - 10^{-6})10^6 = 2\left(1 - \frac{10^{-6}}{2} - \frac{10^{-12}}{2}\right)10^6$$

$$\approx 1.999999 \times 10^6 \text{ light-years.}$$

We *chose* the relative velocity so that the time of the firecracker explosion as observed in the rocket is the same as the time of Caesar's death, namely $t_B' = 0$. The x-coordinate of this explosion is not much different in the two frames because their relative velocity is so small.

d. There exists a frame—the rest frame of the Enterprise—in which Caesar's death and the firecracker explosion occur at the same time. In this frame a signal connecting the two events would have to travel at infinite speed. But this is impossible. Therefore the Enterprise cannot warn Caesar; his death is final. Sorry. (Note: In the language of Chapter 6, the relation between the two events is spacelike, and spacelike events cannot have a cause–effect relationship.)