Review Session #2

Q1) (a) Bra ; Hermitian conjugate : $\langle\psi|\hat{A}^\dagger|\varphi\rangle|\psi\rangle$

(b) Operator ; Hermitian conjugate : $|\psi\rangle\langle\varphi|\hat{A}^\dagger$

(c) wrong question : there is a typing mistake

(d) Ket ; Hermitian conjugate : $\langle\psi|\hat{A}^\dagger|\varphi\rangle\langle\psi| - i\langle\psi|\hat{A}^\dagger$

(e) Operators ; Hermitian Conjugate : $\hat{A}^\dagger|\varphi\rangle\langle\varphi| + i|\psi\rangle\langle\psi|\hat{A}^\dagger$

Q2) $[\hat{A}, \hat{A}^\dagger] = 1$

$\Rightarrow \hat{A}\hat{A}^\dagger - \hat{A}^\dagger\hat{A} = 1$

(a)
$$\begin{aligned}[\hat{A}^\dagger\hat{A}, \hat{A}] &= A^\dagger\hat{A}\hat{A} - \hat{A}\hat{A}^\dagger\hat{A} \\ &= (\hat{A}^\dagger\hat{A} - \hat{A}\hat{A}^\dagger)\hat{A} \\ &= -[\hat{A}, \hat{A}^\dagger]\hat{A} \\ &= -\hat{A}\end{aligned}$$

$$\begin{aligned}[\hat{A}^\dagger\hat{A}, \hat{A}^\dagger] &= \hat{A}^\dagger\hat{A}A^\dagger - \hat{A}^\dagger\hat{A}^\dagger\hat{A} \\ &= \hat{A}^\dagger(\hat{A}\hat{A}^\dagger - \hat{A}^\dagger\hat{A}) \\ &= \hat{A}^\dagger\end{aligned}$$

(b) $\hat{A}|a\rangle = \sqrt{a}\,|a-1\rangle$; $\hat{A}^\dagger|a\rangle = \sqrt{a+1}\,|a+1\rangle$

$\langle a|a'\rangle = \delta_{aa'}$

- $\langle a|\hat{A}|a+1\rangle = \langle a|\sqrt{a+1}|a\rangle = \sqrt{a+1}\langle a|a\rangle$
  $= \sqrt{a+1}$
- $\langle a+1|\hat{A}^\dagger|a\rangle = \sqrt{a+1}\,\langle a+1|a+1\rangle = \sqrt{a+1}$
- $\langle a|A^\dagger A|a\rangle = \sqrt{a}\sqrt{a} = a$
- $\langle a|AA^\dagger|a\rangle = (a+1)$

(e) • $(A+A^\dagger)^2 = A^2 + A^{\dagger 2} + AA^\dagger + A^\dagger A$

$\langle a|(A+A^\dagger)^2|a\rangle = \langle a|A^2|a\rangle + \langle a|A^{\dagger 2}|a\rangle + \langle a|AA^\dagger|a\rangle + \langle a|A^\dagger A|a\rangle$

$= 0 + 0 + (a+1) + a$

$= 2a+1$

• $(A-A^\dagger)^2 = A^2 + A^{\dagger 2} - AA^\dagger - A^\dagger A$

$\therefore \langle a|(A-A^\dagger)^2|a\rangle = \langle a|A^2|a\rangle + \langle a|A^{\dagger 2}|a\rangle - \langle a|AA^\dagger|a\rangle - \langle a|A^\dagger A|a\rangle$

$= -(a+1) - a$

$= -2a-1$

Q3) $\hat{H} = -\alpha \frac{d^2}{dx^2} + 16\alpha x^2$; ~~Ψ~~

(a) $\left(-\alpha \frac{d^2}{dx^2} + 16\alpha x^2\right)\Psi(x) = -\alpha A \frac{d^2}{dx^2}\left(e^{-2x^2}\right) + 16\alpha A x^2 e^{-2x^2}$

$\frac{d(e^{-2x^2})}{dx} = -4xe^{-2x^2}$

$\frac{d^2}{dx^2}(e^{-2x^2}) = -4e^{-2x^2} + 16x^2e^{-2x^2}$

$\therefore \hat{H}\Psi(x) = -\alpha A\left(-4e^{-2x^2} + 16x^2e^{-2x^2}\right) + 16\alpha A x^2 e^{-2x^2}$

$= 4\alpha A e^{-2x^2}$

$\hat{H}\Psi(x) = 4\alpha\,\Psi(x)$

Thus $\Psi(x)$ is an eigenstate of $\hat{H}$ with eigenvalue $= 4\alpha$