LUMS School of Science and Engineering

PHY-201 **Problem Set # 5 - Solutions** October 08, 2012

(a) We are given that $(|1\rangle, |2\rangle, |3\rangle)$ is an orthonormal set, that is $\langle i|j\rangle = \delta_{ij}$ where each of $i$ and $j$ can take on values 1, 2 and 3. The total number of systems is $N = 10^6$, each of which is in the state:

$$|\psi\rangle = \sqrt{\frac{1}{5}}|1\rangle - i\sqrt{\frac{1}{5}}|2\rangle - \sqrt{\frac{3}{5}}|3\rangle \; . \tag{1}$$

It is easily seen that the dual state is:

$$\langle\psi| = \langle 1|\sqrt{\frac{1}{5}} + \langle 2|i\sqrt{\frac{1}{5}} - \langle 3|\sqrt{\frac{3}{5}} \; . \tag{2}$$

Using the orthonormality condition mentioned above, it is easily seen that:

$$\langle\psi|\psi\rangle = (\langle 1|\sqrt{\frac{1}{5}} + \langle 2|i\sqrt{\frac{1}{5}} - \langle 3|\sqrt{\frac{3}{5}} \;\;)\;\; (\sqrt{\frac{1}{5}}|1\rangle - i\sqrt{\frac{1}{5}}|2\rangle - \sqrt{\frac{3}{5}}|3\rangle \;\;) = 1 \; . \tag{3}$$

Now if a measurement process is performed on the system, its state vector (that in (1)) collapses to either of the states $|1\rangle$, $|2\rangle$ or $|3\rangle$. Suppose that it collapses to the state $|i\rangle$. Then the probability of this occurrence (by definition) is:

$$P_{|i\rangle} = \frac{|\langle i|\psi\rangle|^2}{\langle\psi|\psi\rangle} \; . \tag{4}$$

Using the orthonormality condition, it follows that:

$$\langle 1|\psi\rangle = \sqrt{\frac{1}{5}} \;\;,\;\; \langle 2|\psi\rangle = -i\sqrt{\frac{1}{5}} \;\;,\;\; \langle 3|\psi\rangle = -\sqrt{\frac{3}{5}} \; . \tag{5}$$

$$\Rightarrow \qquad |\langle 1|\psi\rangle|^2 = |\langle 2|\psi\rangle|^2 = \frac{1}{5} \;\text{ and }\; |\langle 3|\psi\rangle|^2 = \frac{3}{5} \; . \tag{6}$$

Using (3) and (6) in (4) gives:

$$P_{|1\rangle} = P_{|2\rangle} = \frac{1}{5} \;\text{ and }\; P_{|3\rangle} = \frac{3}{5} \; . \tag{7}$$

Note that the probabilities in (7) sum to 1. Now clearly in the agglomerate ($N = 10^6$), the tentative number of systems in the state $|i\rangle$ should be:

$$n_{|i\rangle} = P_{|i\rangle} \times N \; , \tag{8}$$

and thus utilizing (7) in (8) we get:

$$n_{|1\rangle} = n_{|2\rangle} = 2 \times 10^5 \; , \; n_{|3\rangle} = 6 \times 10^5 \; . \tag{9}$$

Note also that the number of systems in (9) sum to $N = 10^6$. The result found above is indeed tentative rather than exact. This is so because of the fact that we have used probabilities. Of course our claim in (9) becomes increasingly accurate as $N$ becomes large, becoming exact in the limit $N \to \infty$.

(b) The quantity $\langle i|\psi\rangle$ is the usual dot of the vector $|i\rangle$ with $|\psi\rangle$, that is it is the component of $|\psi\rangle$ along the direction $|i\rangle$.

(c) Similarly, $\langle\psi|i\rangle$ is the component of the vector $|i\rangle$ along $|\psi\rangle$.

(d) Here we define the operator $\hat{C}$ in the following way and describe its action:

$$\hat{C}|1\rangle = |2\rangle \ , \ \hat{C}|2\rangle = |3\rangle \ , \ \hat{C}|3\rangle = |1\rangle \ . \tag{10}$$

This operator cyclically interchanges the vectors $(|1\rangle, |2\rangle, |3\rangle)$. In particular, it maps $|1\rangle$ to $|2\rangle$ and $|2\rangle$ to $|3\rangle$. Finally, $|3\rangle$ is mapped back to $|1\rangle$.

(e) Next lets examine the action of $\hat{C}$ on $|\psi\rangle$. This is accomplished by the use of (1) and (10):

$$|\phi\rangle = \hat{C}|\psi\rangle = \hat{C}(\sqrt{\frac{1}{5}}|1\rangle - i\sqrt{\frac{1}{5}}|2\rangle - \sqrt{\frac{3}{5}}|3\rangle) = \sqrt{\frac{1}{5}}\hat{C}|1\rangle - i\sqrt{\frac{1}{5}}\hat{C}|2\rangle - \sqrt{\frac{3}{5}}\hat{C}|3\rangle$$

$$= \sqrt{\frac{1}{5}}|2\rangle - i\sqrt{\frac{1}{5}}|3\rangle - \sqrt{\frac{3}{5}}|1\rangle = -\sqrt{\frac{3}{5}}|1\rangle + \sqrt{\frac{1}{5}}|2\rangle - i\sqrt{\frac{1}{5}}|3\rangle \ . \tag{11}$$

We can now use (2) and (11) to compute $\langle\psi|\phi\rangle$:

$$\langle\psi|\phi\rangle = (\ \langle 1|\sqrt{\frac{1}{5}} + \langle 2|i\sqrt{\frac{1}{5}} - \langle 3|\sqrt{\frac{3}{5}}\ )\ (-\sqrt{\frac{3}{5}}|1\rangle + \sqrt{\frac{1}{5}}|2\rangle - i\sqrt{\frac{1}{5}}|3\rangle\ )$$

$$-\frac{\sqrt{3}}{5} + \frac{i}{5} + \frac{i\sqrt{3}}{5} = \frac{-\sqrt{3} + i(1+\sqrt{3})}{5} \ . \tag{12}$$

And similarly it can be shown that:

$$\langle\phi|\psi\rangle = \frac{-\sqrt{3} - i(1+\sqrt{3})}{5} \ . \tag{13}$$

It is worth noting that the expressions found in (12) and (13) are complex conjugates of each other, as expected.

(f) Let $\hat{I}$ be the familiar identity operator. In particular, its action on the kets $(|1\rangle, |2\rangle, |3\rangle)$ is given by:

$$\hat{I}|1\rangle = |1\rangle \ , \ \hat{I}|2\rangle = |2\rangle \ , \ \hat{I}|3\rangle = |3\rangle \ . \tag{14}$$

We define a new operator given by:

$$\hat{D} = \sqrt{\frac{2}{3}}\hat{I} + i\sqrt{\frac{1}{3}}\hat{C} \ . \tag{15}$$

And it is a straight forward exercise to utilize (10) and (14) in (15) to determine the action of $\hat{D}$ on each of the vectors $(|1\rangle, |2\rangle, |3\rangle)$:

$$\hat{D}|1\rangle = \sqrt{\frac{2}{3}}|1\rangle + i\sqrt{\frac{1}{3}}|2\rangle \ , \ \hat{D}|2\rangle = \sqrt{\frac{2}{3}}|2\rangle + i\sqrt{\frac{1}{3}}|3\rangle \ , \ \hat{D}|3\rangle = i\sqrt{\frac{1}{3}}|1\rangle + \sqrt{\frac{2}{3}}|3\rangle \ . \tag{16}$$

(g) The matrix elements of $\hat{D}$ can be computed simply by operating with $\langle i|$ $(i = 1, 2, 3)$ on the left hand sides of (16):

$$\hat{D} \ = \ \begin{pmatrix} \langle 1|\hat{D}|1\rangle & \langle 1|\hat{D}|2\rangle & \langle 1|\hat{D}|3\rangle \\ \langle 2|\hat{D}|1\rangle & \langle 2|\hat{D}|2\rangle & \langle 2|\hat{D}|3\rangle \\ \langle 3|\hat{D}|1\rangle & \langle 3|\hat{D}|2\rangle & \langle 3|\hat{D}|3\rangle \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{2}{3}} & 0 & i\sqrt{\frac{1}{3}} \\ i\sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} & 0 \\ 0 & i\sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} \end{pmatrix} \ . \tag{17}$$

Note that in particular $D_{12} \neq D^\star_{21}$ and this alone is sufficient to nullify the following claim:

$$D_{ij} = D^\star_{ji} \ \ \forall \ \ i,j = 1,2,3 \ . \tag{18}$$

(h) Here all that we require is an examination of the action of $\hat{C}^3$ on an arbitrary state $|\lambda\rangle = \lambda_1|1\rangle + \lambda_2|2\rangle + \lambda_3|3\rangle$:

$$\hat{C}^3|\lambda\rangle = \hat{C}^3(\ \lambda_1|1\rangle + \lambda_2|2\rangle + \lambda_3|3\rangle\ ) = \hat{C}^2(\ \lambda_1|2\rangle + \lambda_2|3\rangle + \lambda_3|1\rangle\ )$$

$$= \hat{C}(\ \lambda_1|3\rangle + \lambda_2|1\rangle + \lambda_3|2\rangle\ ) = \ \lambda_1|1\rangle + \lambda_2|2\rangle + \lambda_3|3\rangle \ = |\lambda\rangle = 1.|\lambda\rangle$$

Thus for the operator $\hat{C}$, all states are eigenstates and the corresponding eigenvalue is 1: the only eigenvalue.

(i) Consider now the operator given by:

$$\hat{R}|1\rangle = \cos\theta|1\rangle + \sin\theta|2\rangle\ ,\ \hat{R}|2\rangle = -\sin\theta|1\rangle + \cos\theta|2\rangle\ ,\ \hat{R}|3\rangle = |3\rangle\ . \tag{19}$$

The action of this operator can be best described after having constructed its matrix representation. Refer to part (k) below for the answer.

(j) The action of an operator on a linear combination of states, as well as the dotting of two vectors has been adequately illustrated so far. Following the outlined procedure, we can show that:

$$|\phi\rangle = \hat{R}|\psi\rangle = e^{i\theta}(\ \sqrt{\frac{1}{5}}|1\rangle - i\sqrt{\frac{1}{5}}|2\rangle\ ) - \sqrt{\frac{3}{5}}|3\rangle\ , \tag{20}$$

$$\langle\psi|\phi\rangle = \frac{2}{5}\ e^{i\theta} + \frac{3}{5} \quad \text{and} \quad \langle\phi|\psi\rangle = \frac{2}{5}\ e^{-i\theta} + \frac{3}{5}\ . \tag{21}$$

(k) Following a procedure similar to the one adopted in part (g), the matrix representation of $\hat{R}$ can be shown to be:

$$\hat{R} \ = \ \begin{pmatrix} \langle 1|\hat{R}|1\rangle & \langle 1|\hat{R}|2\rangle & \langle 1|\hat{R}|3\rangle \\ \langle 2|\hat{R}|1\rangle & \langle 2|\hat{R}|2\rangle & \langle 2|\hat{R}|3\rangle \\ \langle 3|\hat{R}|1\rangle & \langle 3|\hat{R}|2\rangle & \langle 3|\hat{R}|3\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\ . \tag{22}$$

Note that in particular $R_{12} \neq R^\star_{21}$ and this alone is sufficient to nullify the following claim:

$$R_{ij} = R^\star_{ji} \ \ \forall \ \ i,j = 1,2,3 \ . \tag{23}$$

Finally, let us interpret what the operator $\hat{R}$ represents: if we were to identify the vectors $|1\rangle,\ |2\rangle,\ |3\rangle$ with $\hat{i},\ \hat{j},\ \hat{k}$ respectively, then the matrix in (22) represents a clockwise rotation of coordinates by angle $\theta$ about the $z$ axis.

(l) Finally, the eigenvalues can be calculated as follows:

$$\hat{R}|\lambda\rangle = \lambda|\lambda\rangle \Rightarrow (\hat{R} - \lambda\hat{I})|\lambda\rangle = |\theta\rangle\ . \tag{24}$$

The fact that we are looking for a nontrivial ket in (24) requires us to impose the condition:

$$|\hat{R} - \lambda\hat{I}| = 0\ . \tag{25}$$

From here it is a straight forward exercise to set up the characteristic equation to get the eigen values:

$$\lambda_1 = 1\ ,\ \lambda_2 = e^{i\theta}\ ,\ \lambda_3 = e^{-i\theta}\ . \tag{26}$$

This completes the solution.