Problem Set # 10

Q1)

$$U_\theta|0\rangle = \cos(\theta/2)\,|0\rangle + \sin\theta/2\,|1\rangle$$
$$U_\theta|1\rangle = -\sin\theta/2\,|0\rangle + \cos\theta/2\,|1\rangle$$

Therefore $U_\theta = \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix}$

Our goal is to write U_θ in term of exponential that involves θ and the pauli matrices.

Lets check: $\exp\left(-i\frac{\theta}{2}\delta_y\right)$

$$\exp\left(\frac{i\theta}{2}\delta_y\right) = 1 + \left(\frac{-i\theta}{2}\right)\delta_y + \frac{1}{2!}\left(\frac{-i\theta}{2}\right)^2\delta_y^2 + \frac{1}{3!}\left(\frac{-i\theta}{2}\right)^3\delta_y^3$$
$$+ \frac{1}{4!}\left(\frac{-i\theta}{2}\right)^4\delta_y^4 + \ldots \quad \text{— (1)}$$

Note that $\delta_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$$\therefore \delta_y^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}$$

$$\therefore \exp\left(\frac{i\theta}{2}\delta_y\right) = \mathbb{1} + \left(\frac{i\theta}{2}\right)\delta_y - \frac{1}{2!}\left(\frac{\theta}{2}\right)^2\mathbb{1} + \frac{1}{3!}\left(\frac{\theta}{2}\right)^3\delta$$
$$+ \frac{1}{4!}\left(\frac{\theta}{2}\right)^4\mathbb{1} + \ldots$$
$$= \left(1 - \frac{1}{2!}\left(\frac{\theta}{2}\right)^2 + \frac{1}{4!}\left(\frac{\theta}{2}\right)^4 + \ldots\right)\mathbb{1} - i\delta_y\left[\frac{\theta}{2} - \frac{1}{3!}\left(\frac{\theta}{2}\right)^3 + \right.$$

$\because \exp\left(-i\frac{\theta}{2}\sigma_y\right) = \cos\frac{\theta}{2}\,\mathbb{1} - i\sin\frac{\theta}{2}\,\sigma_y$

$= \cos\frac{\theta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i\sin\frac{\theta}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$= \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ +\sin\theta/2 & \cos\theta/2 \end{pmatrix}$

$= U_\theta$

Therefore $\boxed{U_\theta = \exp\left(-i\frac{\theta}{2}\sigma_y\right)}$

Q2) (a) $|\psi\rangle = \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|11\rangle$

$= \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right) \otimes |1\rangle$

Therefore $|\psi\rangle$ is not entangled.

(b) let $|\psi\rangle = \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle$ — (i)

Let $|\psi\rangle = |\phi\rangle \otimes |\chi\rangle$

where $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$

and $|\chi\rangle = \gamma|0\rangle + \Omega|1\rangle$

$|\psi\rangle = (\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \Omega|1\rangle)$

$= \alpha\gamma|00\rangle + \beta\gamma|10\rangle + \alpha\Omega|01\rangle + \beta\Omega|11\rangle$

— (ii)

equate (I) and (II),

$$\frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle = \alpha\gamma|00\rangle + \beta\gamma|10\rangle + \alpha\Omega|01\rangle + \beta\Omega|11\rangle$$

$\therefore$
$$\alpha\gamma = 0 \quad \text{— (a)}$$
$$\beta\gamma = \frac{1}{\sqrt{2}} \quad \text{— (b)}$$
$$\alpha\Omega = -1/\sqrt{2} \quad \text{— (c)}$$
$$\beta\Omega = 0 \quad \text{— (d)}$$

from (a), either $\alpha = 0$ or $\gamma = 0$
but from (b), $\gamma \neq 0 \Rightarrow \alpha = 0$.
but if $\alpha = 0$, then eq (c) does not hold.

Therefore there is no possible solution for eq (a), (b), (c) and (d). Thus eq (II) does not hold true.

Therefore $|\psi\rangle$ is entangled.

Q3) (a) $|\Psi\rangle = \sqrt{\frac{1}{6}}|000\rangle + \frac{1}{\sqrt{6}}|010\rangle + \sqrt{\frac{2}{3}}|111\rangle$

If I get $|0\rangle$ after measuring first qubit, then the state ~~after~~ will collapse to $|\phi\rangle$, where $|\phi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |010\rangle)$

If I get $|1\rangle$ after measuring first qubit, then the collapsed state will be $|\phi\rangle$, where
$|\phi\rangle = |111\rangle$

This is entangled state.

(b) If I get $|0\rangle$ after measuring second qubit, then the state will collapse to $|\phi\rangle$, where $|\phi\rangle = |000\rangle$

And if I get $|1\rangle$, then the collapsed state will be $|\phi\rangle = \sqrt{\frac{1}{5}}\,|010\rangle + \sqrt{\frac{4}{5}}\,|111\rangle$

If the measurement of the 3rd qubit yields $|0\rangle$, then the state will collapse to
$|\phi\rangle = \frac{1}{\sqrt{2}}|000\rangle + \frac{1}{\sqrt{2}}|010\rangle$

And if I get $|1\rangle$, then the collapsed state will be $|\phi\rangle = |111\rangle$

Q4) (a) $|0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$; $|1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$\therefore |0\rangle\langle 0| - e^{i\theta}|1\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & -e^{i\theta} \end{pmatrix}$

(b) $|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$; $|10\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$; $|01\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$; $|11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$$|00\rangle\langle 00| = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$|11\rangle\langle 11| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$|00\rangle\langle 00| - e^{i\theta}|11\rangle\langle 11| = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -e^{i\theta} \end{pmatrix}$$

Q5) Before solving this question, consider a simple case.

Consider a polarized beam of ~~ligh~~ photons in a state $|\psi\rangle = \cos\theta\,|0\rangle + \sin\theta\,|1\rangle$

If there were N photons in the beam, then on average $\cos^2\theta\, N$ will be in the state $|0\rangle$ and $\sin^2\theta\, N$ will be in the state $|1\rangle$.

If I now keep a polarizer with axis along $\hat{x}$ axis, then the photon in $|0\rangle$ will survive. Therefore after the polarizer, state will be $|0\rangle$

and only $\cos^2\theta\, N$ photons will be there in the beam.

Since intensity is proportional to the # of photons, so we can say that the intensity of the beam decreases by a factor of $\cos^2\theta$, after passing through a polarizer aligned at an angle θ from the polarization axis.

What if we do not have a polarized light, but an unpolarized light. Then there is no constant value of θ. To find the transmitted intensity, we need to find the average of $\cos^2\theta$, which is $\frac{1}{2}$.

$\therefore$ After ~~pot~~ passing through a polarizer, the intensity of the unpolarized light ~~get~~ ~~ben~~ decreases by 50%.

Now lets try to answer the question using the above information.

(a)

Unpolarized light	x-polarized light	
Intensity $= I_0$	$I = \frac{I_0}{2}$	$I = \cos^2\theta \frac{I_0}{2}$
	first filter	2^{nd} filter at angle θ

Therefore after passing through the 2^{nd} filter, the intensity decreases by $\frac{I}{I_0} = \frac{1}{2}\cos^2\theta$

(b) $I = \frac{1}{2} I\cos^2\theta \, I_0$ | $I = \frac{1}{2} I_0 \cos^4\theta$

3^{rd} polarizer at angle θ

Thus $\frac{I}{I_0} = \frac{1}{2}\cos^4\theta$

Q6) (a) $C_{NOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

Suppose that C_{NOT} can be written as a tensor product of 2 one qubit gates, i.e $\hat{C}_{NOT} = \hat{A} \otimes \hat{B}$, where

$\hat{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\hat{B} \begin{pmatrix} e & f \\ g & h \end{pmatrix}$

$$A \otimes B = \begin{pmatrix} a(\hat{B}) & b(\hat{B}) \\ c(\hat{B}) & d(\hat{B}) \end{pmatrix}$$

if $C_{NOT} = \hat{A} \times \hat{B}$, then

$$\hat{O} = c\hat{B} = b\hat{B}$$

this means that ~~if~~ either $c = b = 0$, or $\hat{B} = 0$.

If $\hat{B} = 0$, then $\hat{A} \times \hat{B} = 0$ and hence $C_{NOT} \neq \hat{A} \otimes \hat{B}$ and hence it is verified that C_{NOT} can not be written as a tensor product of 2 single qubit gate.

What if $\hat{B} \neq 0$ and $c = b = 0$? Then

$$a\hat{B} = \mathbb{1} \implies \hat{B} = \frac{1}{a}\mathbb{1}.$$

and hence $d\hat{B} = \frac{d}{a}\mathbb{1} \Rightarrow$ but this does not satisfy $d\hat{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Thus it is verified that $C_{NOT} \neq \hat{A} \otimes \hat{B}$.

(b) ~~$|00\rangle$~~ $|0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

~~$|00\rangle\langle +$~~ ~~$|00\rangle\langle 0$~~ $|0\rangle\langle 0| \otimes \mathbb{1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$$|1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\hat{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$|1\rangle\langle 1| \otimes \hat{X} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$|0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \hat{X} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(c) $S|00\rangle = |00\rangle$
$S|01\rangle = |10\rangle$
$S|10\rangle = |01\rangle$
$S|11\rangle = |11\rangle$

The above relations shows that $\hat{S}$ maps $|ab\rangle$ to $|ba\rangle$. In other words, It swaps.

$F|000\rangle = |000\rangle$; $F|001\rangle = |001\rangle$; $F|010\rangle = |010\rangle$; $F|011\rangle = |01$
$F|100\rangle = |100\rangle$; $F|101\rangle = |110\rangle$; $F|110\rangle = |101\rangle$; $F|111\rangle = |111\rangle$

Therefore, it is now obvious that if first qubit is in $|0\rangle$, then F will not do anything to the other two qubits. Otherwise, it will swap them.

Q7) $U_f |x, y\rangle = |x, y \oplus f(x)\rangle$ (Given)

$$U_f \left(\frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x, 0\rangle \right) = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} U_f |x, 0\rangle$$

* This is because of the linearity of the linear operators.

$$= \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x, 0 \oplus f(x)\rangle$$

$$= \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x, f(x)\rangle$$