LUMS School of Science and Engineering

PHY201 Modern Physics – Assignment # 7,8 (double credit)

Due on 4:00pm, Friday, Nov 9

1. Consider a particle which is confined to move along the positive x-axis and whose Hamiltonian is $\hat{H} = \mathcal{E}d^2/dx^2$, where $\mathcal{E}$ is a positive real constant having the dimensions of energy.

   (a) Find the wave function that corresponds to an energy eigenvalue of $9\mathcal{E}$ (make sure that the function you find is finite everywhere along the positive x-axis and is square integrable). Normalize this wave function.

   (b) Calculate the probability of finding the particle in the region $0 \le x \le 15$.

   (c) Is the wave function derived in (a) an eigenfunction of the operator $\hat{A} = d/dx - 7$?

   (d) Calculate the commutator $[\hat{H}, \ \hat{A}]$.

2. Consider the wave functions:

$$\psi(x, y) = \sin 2x \ \cos 5x, \qquad \phi(x, y) = e^{-2(x^2+y^2)}, \qquad \chi(x, y) = e^{-i(x+y)}.$$

   (a) Verify if any of the wave functions is an eigenfunction of $\hat{A} = \partial/\partial x + \partial/\partial y$.

   (b) Find out if any of the wave functions is an eigenfunction of $\hat{B} = \partial^2/\partial x^2 + \partial^2/\partial y^2 + 1$.

   (c) Calculate the actions of $\hat{A}\hat{B}$ and $\hat{B}\hat{A}$ on each one of the wave functions and infer $[\hat{A}, \ \hat{B}]$.

3. A particle of mass m, which moves freely inside an infinite potential well of length a, is initially in the state $\psi(x, 0) = \sqrt{3/5a}\,\sin(3\pi x/a) + (1/\sqrt{5a})\,\sin(5\pi x/a)$.

   (a) Find $\psi(x, t)$ at any later time t.

   (b) Calculate the probability density $\rho(x, t)$.

4. A system is initially in the state $|\psi_0\rangle = [\sqrt{2}|\phi_1\rangle + \sqrt{3}|\phi_2\rangle + |\phi_3\rangle + |\phi_4\rangle]/\sqrt{7}$, where $|\phi_n\rangle$ are eigenstates of the system's Hamiltonian such that $\hat{H}|\phi_n\rangle = n^2\mathcal{E}_0|\phi_n\rangle$.

   (a) If energy is measured, what values will be obtained and with what probabilities?

   (b) Consider an operator $\hat{A}$ whose action on $|\phi_n\rangle$ is defined by $\hat{A}|\phi_n\rangle = (n+1)a_0|\phi_n\rangle$. If A is measured, what values will be obtained and with what probabilities?

   (c) Suppose that a measurement of the energy yields $4\mathcal{E}_0$. If we measure A immediately afterwards, what value will be obtained?

5. Consider a hydrogen atom which is in its ground state; the ground state wave function is given by

$$\Psi(r, \theta, \varphi) = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0},$$

   where a_0 is the Bohr radius.

   (a) Find the most probable distance between the electron and the proton when the hydrogen atom is in its ground state.

   (b) Find the average distance between the electron and the proton.

6. Consider a muonic atom which consists of a nucleus that has Z protons (no neutrons) and a negative muon moving around it; the muon's charge is $-e$ and its mass is 207 times the mass of the electron, $m_{\mu^-} = 207 m_e$. For a muonic atom with $Z = 6$, calculate

(a) the radius of the first Bohr orbit,

(b) the energy of the ground, first, and second excited states.

7. (a) Calculate the expectation value $\langle r \rangle_{21}$ for the hydrogen atom and compare it with the value r at which the radial probability density reaches its maximum for the state $n = 2, l = 1$.

(b) Calculate the width of the probability density distribution for r.

8. An electron in a hydrogen atom is in the energy eigenstate

$$\psi_{2,1,-1}(r, \theta, \varphi) = N r e^{-r/2a_0} Y_{1,-1}(\theta, \varphi).$$

(a) Find the normalization constant, N.

(b) What is the probability per unit volume of finding the electron at $r = a_0$, $\theta = 45°$, $\varphi = 60°$?

(c) What is the probability per unit radial interval (dr) of finding the electron at $r = 2a_0$? (One must take an integral over θ and φ at $r = 2a_0$.)

(d) If the measurements of $\hat{L}^2$ and $\hat{L}_z$ were carried out, what will be the results?

9. The wave function of a hydrogen-like atom at time $t = 0$ is

$$\Psi(\vec{r}, 0) = \frac{1}{\sqrt{11}} \left[\sqrt{3}\psi_{2,1,-1}(\vec{r}) - \psi_{2,1,0}(\vec{r}) + \sqrt{5}\psi_{2,1,1}(\vec{r}) + \sqrt{2}\psi_{3,1,1}(\vec{r}) \right],$$

where $\psi_{nlm}(\vec{r})$ is a normalized eigenfunction (i.e., $\psi_{nlm}(\vec{r}) = R_{nl}(r) Y_{lm}(\theta, \varphi)$).

(a) What is the time-dependent wave function?

(b) If a measurement of energy is made, what values could be found and with what probabilities?

(c) What is the probability for a measurement of $\hat{L}_z$ which yields $-1\hbar$?

10. Calculate $\Delta r \Delta p_r$ with respect to the state

$$\psi_{2,1,0}(\vec{r}) = \frac{1}{\sqrt{6}} \left(\frac{1}{a_0} \right)^{3/2} \frac{r}{2a_0} e^{-r/2a_0} \, Y_{1,0}(\theta, \varphi),$$

and verify that $\Delta r \Delta p_r$ satisfies the Heisenberg uncertainty principle.