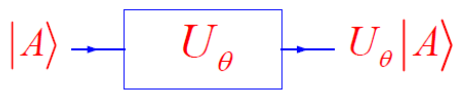
LUMS School of Science and Engineering

PHY201 Modern Physics – Assignment # 10

Due at 4:00pm, Friday, 7 Dec 2012

Q.1 Consider the following one-bit gate. In terms of the Pauli operators $\{1, \sigma_x, \sigma_y, \sigma_z\}$, find a realization of the unitary operator U which can implement the gate. In other words, write U as an exponential that involves θ and the Pauli operators.

One-bit gate:



$$U_\theta|0\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle$$
$$U_\theta|1\rangle = \cos(\theta/2)|1\rangle - \sin(\theta/2)|0\rangle$$

Q.2 a) Show that the state $\frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|11\rangle$ is unentangled.

b) Show that the state $\frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle$ is entangled.

Q.3 The 3-qubit register of a quantum computer is in the state,

$$|\Psi\rangle = \sqrt{\frac{1}{6}}\,|000\rangle + \sqrt{\frac{1}{6}}\,|010\rangle + \sqrt{\frac{2}{3}}\,|111\rangle$$

a) Into what state will $|\Psi\rangle$ collapse after a measurement of the first qubit? Is it entangled?

b) Repeat for the second and third qubits.

Q.4 a) You can think of $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$ as operators. If $|0\rangle$ is represented by $\begin{bmatrix}1\\0\end{bmatrix}$ and if $|1\rangle$ is represented by $\begin{bmatrix}0\\1\end{bmatrix}$, how would you represent $|0\rangle\langle 0| - e^{i\theta}|1\rangle\langle 1|$?

b) Now let's go to two qubits. Let $|00\rangle \equiv |0\rangle_a \otimes |0\rangle_b$ etc, and $\langle 00| \equiv_b \langle 0| \otimes_a \langle 0|$. How will you represent $|00\rangle\langle 00| - e^{i\theta}|11\rangle\langle 11|$?

Q.5 Let $|0\rangle$ denote the state of a photon that is polarized along the x-axis and $|1\rangle$ denote the state of a photon that is polarized along the y-axis. Now suppose that we have a rotated coordinate and define new states that are linear combinations of the old states:

$$|A\rangle = |0\rangle\cos\theta + |1\rangle\sin\theta \quad \text{and} \quad |B\rangle = -|0\rangle\sin\theta + |1\rangle\cos\theta.$$

a) Suppose that a beam of photons that is randomly polarized is passed through a filter that allows only x-polarized photons to pass through it. Now suppose that the first filter is followed by a second filter that is tilted at angle θ. What is the intensity of light that passed through the second one?

b) The light from the second filter goes through a third filter that is parallel to the first one. What is the intensity of light that emerges from this?

Q.6 a) Verify that the C_{not} gate transformation, defined as below,

$$C_{not}: \begin{array}{ccc} |00\rangle & \to & |00\rangle \\ |01\rangle & \to & |01\rangle \\ |10\rangle & \to & |11\rangle \\ |11\rangle & \to & |10\rangle \end{array} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

cannot be written as a product of two single-bit transformations.

b) Verify the following: $C_{not} = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X.$

c) By applying F and S on to suitable input states, justify the names "controlled swap" and "swap" for the following two operations:

$$F = |0\rangle\langle 0| \otimes I \otimes I + |1\rangle\langle 1| \otimes S$$

$$S = |00\rangle\langle 00| + |01\rangle\langle 10| + |10\rangle\langle 01| + |11\rangle\langle 11|.$$

Q.7 Carefully verify:

$$\begin{aligned} U_f\left(\frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}|x,0\rangle\right) &= \frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}U_f(|x,0\rangle) \\ &= \frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}|x,\, f(x)\rangle \end{aligned}$$

This formula lies at the heart of why quantum computers (when they become operational in a few decades from now) will be so much more powerful than any digital computer.