The AdS/CFT correspondence

Senior Project Report

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Studying strongly coupled gauge theories using the AdS/CFT correspondence

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Foreword

To understand the behavior of strongly coupled gauge theories has been a long standing problem. Since the relevant coupling constants are large a perturbation expansion is no longer workable for these problems. An alternate approach must therefore be formulated. Fortunately, the AdS/CFT correspondence conjecture offers a way out. This correspondence is the main subject of this work.

In an attempt to make this thesis as self-contained as possible, a detailed discussion of conformal field theories and anti de sitter spaces, the two main halves of the correspondence, has been included in the first two chapters. The analysis of gauge theories in the dual gravity picture relies heavily on the concept of D-branes which is the subject of chapter 3. Chapter 4 provides motivation for how the AdS/CFT correspondence comes about and chapters 5 and 6 demonstrate how the correspondence may be applied to studying magnetic catalysis of chiral symmetry breaking and quark confinement/deconfinement transitions.
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Chapter 1

Conformal Field Theories

This chapter relies heavily on the discussion in chapter 1 of [1]. To avoid digression, some details will be skipped and the interested reader is referred to [1] to fill them in. We will begin by looking at some essential properties of conformal field theories in flat space in an attempt to develop an understanding of these theories before discussing the AdS/CFT correspondence.

1.1 Introduction

A conformal field theory is a quantum field theory whose lagrangian is invariant under conformal transformations. Conformal transformations are transformations that locally preserve the angle between any two lines defined on a space. Mathematically this translates into the following:

\[ g'_{\rho\sigma}(x') \frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} = \Lambda(x) g_{\mu\nu}(x) \]  

For the flat space with constant metric case, this becomes:

\[ \eta_{\rho\sigma} \frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} = \eta_{\mu\nu} \]  

Its an easy exercise to see how conformal transformations preserve angles. Consider the following: By the Cauchy-Schwarz equation, we know that:

\[ \cos \theta = \frac{A \cdot B}{|A||B|} \]  

Now,
Under a coordinate transformation, this becomes:

\[
\Lambda(\epsilon) \eta_{\mu\nu} \Lambda(\epsilon)^{-1} = \eta_{\mu\nu} + \partial_\nu \epsilon_\mu + \partial_\mu \epsilon_\nu + O(\epsilon^2)
\]
Some useful relations:

Let us derive some useful relations to be used later on. Let’s differentiate eq on both sides:

\[ \partial^\nu (\partial_\mu \epsilon^\nu + \partial_\nu \epsilon_\mu) = \frac{2}{d} \partial^\nu (\partial_\nu \epsilon) \eta_{\mu \nu} \]  
(1.7)

\[ \Rightarrow \partial_\mu (\partial_\nu \epsilon) + \Box \epsilon_\mu = \frac{2}{d} \partial_\mu (\partial_\nu \epsilon) \]  
(1.8)

\[ \Rightarrow \partial_\mu \partial_\nu (\partial_\nu \epsilon) + \Box \partial_\nu \epsilon_\mu = \frac{2}{d} \partial_\mu \partial_\nu (\partial_\nu \epsilon) \]  
(1.9)

where \( \Box = \partial_\mu \partial^\mu \) and in the last step, we acted with \( \partial_\nu \) on both sides. Now, we interchange \( \mu \leftrightarrow \nu \) and add the expression obtained to the previous equation. We get:

\[ 2 \partial_\mu \partial_\nu (\partial_\nu \epsilon) + \Box (\frac{2}{d} (\partial_\nu \epsilon) \eta_{\mu \nu}) = \frac{4}{d} \partial_\mu \partial_\nu (\partial_\nu \epsilon) \rightarrow (\eta_{\mu \nu} \Box + (d-2) \partial_\mu \partial_\nu (\partial_\nu \epsilon) = 0 \]  
(1.10)

Now, we can multiply \( \eta^{\mu \nu} \) on both sides and obtain:

\[ (d-1) \Box (\partial_\nu \epsilon) = 0 \]  
(1.11)

The second important expression we will need to use later is the following: If we act with \( \partial_\rho \) on both sides of (1.9) and then permute the indices cyclicly, we obtain the following three equations:

\[ \partial_\rho \partial_\mu \epsilon_\nu + \partial_\rho \partial_\nu \epsilon_\mu = \frac{2}{d} \eta_{\mu \nu} \partial_\rho (\partial_\nu \epsilon) \]  
(1.12)

\[ \partial_\nu \partial_\rho \epsilon_\mu + \partial_\mu \partial_\rho \epsilon_\nu = \frac{2}{d} \eta_{\rho \mu} \partial_\nu (\partial_\nu \epsilon) \]  
(1.13)

\[ \partial_\mu \partial_\nu \epsilon_\rho + \partial_\nu \partial_\mu \epsilon_\rho = \frac{2}{d} \eta_{\nu \rho} \partial_\mu (\partial_\nu \epsilon) \]  
(1.14)

Now, if we subtract the (1.12) from the sum of (1.13) and (1.14), we get:

\[ 2 \partial_\mu \partial_\nu \epsilon_\rho = \frac{2}{d} (-\eta_{\mu \nu} \partial_\rho + \eta_{\rho \mu} \partial_\nu + \eta_{\nu \rho} \partial_\mu) (\partial_\nu \epsilon) \]  
(1.15)
1.3 Conformal Group in $d \geq 3$

Notice that (1.11) tells us that $(\partial.\epsilon)$ can be, at most, linear in $x^\mu$, that $(\partial.\epsilon) = C + D_\mu x^\mu$ where $C$ and $D_\mu$ are constants. This allows us to make the following ansatz for $\epsilon_\mu$:

$$\epsilon_\mu = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho$$  \hspace{1cm} (1.16)

where we impose that the constants $a_\mu, b_{\mu\nu}, c_{\mu\nu\rho} \ll 1$ and $c_{\mu\nu\rho} = c_{\mu\rho\nu}$. Using this, we now study the individual terms in (1.16) and see what they lead us to.

- The term $a_\mu$, we know, describes an infinitesimal translation for which the generator is $P_\mu = -i\partial_\mu$

- To study the term linear in $x$, we plug (1.16) in (1.11) which gives us

$$b_{\nu\mu} + b_{\mu\nu} = \frac{2}{d}(\eta^{\rho\sigma} b_{\rho\sigma} \eta_{\mu\nu})$$  \hspace{1cm} (1.17)

From this expression, we notice that $b_{\mu\nu}$ can be split into a symmetric and an antisymmetric part as follows:

$$b_{\mu\nu} = \alpha \eta_{\mu\nu} + m_{\mu\nu}$$  \hspace{1cm} (1.18)

where $\alpha = \frac{(\eta^{\rho\sigma} b_{\rho\sigma}) \eta_{\mu\nu}}{d}$ and $m_{\mu\nu} = m_{\nu\mu}$

The symmetric term describes infinitesimal transformations of the following form: $x'^\mu = (1 + \alpha) x^\mu$ generated by $D = -i x^\mu \partial_\mu$. The antisymmetric part $m_{\mu\nu}$ corresponds to infinitesimal rotations $x'^\mu = (\delta_\mu^\nu + m_{\mu\nu}) x^\nu$ with the generator being the familiar angular momentum operator $L_{\mu\nu} = i (x_\mu \partial_\nu - x_\nu \partial_\mu)$

- The quadratic term can be studied by inserting eq (1.16) into expression (1.15). We can then calculate

$$\partial.\epsilon = \partial^\mu \epsilon_\mu = \partial^\mu (a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho)$$

$$\implies b_{\mu\nu} \delta^{\mu\nu} + c_{\mu\nu\rho} (x^\rho \delta^{\mu\nu} + \delta^{\rho\mu} x^{\nu})$$

$$\implies b_{\mu} + c_{\mu\rho} x^\rho + c_{\nu\rho} x^\nu$$

Now in the last term switching the dummy indices $\nu \leftrightarrow \rho$ and replacing $\nu \leftrightarrow \mu$ we get:
1.3. CONFORMAL GROUP IN $D \geq 3$

$$\Rightarrow \partial. \epsilon = b^\mu_\mu + 2c^\mu_\mu x^\rho$$

Acting with $\partial_\nu$:

$$\Rightarrow \partial_\nu(\partial. \epsilon) = 2c^\mu_\mu \delta^\rho_\nu \rightarrow 2c^\mu_\mu$$

(1.19)

Using this, in eq (1.15):

$$2\partial_\mu \partial_\nu \epsilon_\rho = \frac{2}{d}(-\eta_\mu_\nu b_\rho + n_\mu_\rho b_\nu + \eta_\nu_\rho b_\mu) \rightarrow 2\epsilon^\mu_\mu$$

$$\Rightarrow \frac{2}{d}(-\eta_\mu_\nu (2c^\mu_\mu) + n_\mu_\rho (2c^\mu_\mu) + \eta_\nu_\rho (2c^\mu_\mu))$$

Now if we explicitly evaluate the left hand side of the equation, we get the following:

$$c_\rho_\mu_\nu = -\eta_\mu_\nu b_\rho + n_\mu_\rho b_\nu + \eta_\nu_\rho b_\mu$$

(1.20)

Where $b_\mu = \frac{1}{d}c^\mu_\rho$.

The change in $x$, $\Delta x$ brought about by this part of $\epsilon$ is given by:

$$\Delta x = (\eta_\mu_\rho b_\nu + n_\mu_\rho b_\nu - \eta_\nu_\rho b_\mu) x^\nu x^\rho \Rightarrow x^\nu x_\mu b_\nu + x_\mu x_\rho b_\rho - x_\rho x_\mu b_\mu$$

$$\Rightarrow 2(x.b)x^\mu - (x.x)b^\mu$$

The resulting transformation is what is called Special Conformal Transformation. Because it comprises of a dilation and a translation, it is straightforward to see that the generator is given by:

$$K_\mu = -\iota(2x_\mu x^\nu \partial_\nu - (x.x)\partial_\mu)$$

(1.21)

The finite form of this transformation is given by:

$$x'^\mu = \frac{x^\mu - (x.x)b^\mu}{1 - 2(b.x) + (b.b)(x.x)}$$

(1.22)

In order to see why this really is the correct form, let us expand the denominator for small values of $b$, using binomial expansion.

That gives us:
$x'^\mu \approx x^\mu + 2(b.x)x^\mu - (x.x)b^\mu + ...$

For future reference, let us list all the conformal transformations with their respective generators here:

- **Translation**:
  
  \[
  x'^\mu = x^\mu + a^\mu \\
  P_\mu = -i \partial_\mu 
  \]

- **Dilation**:
  
  \[
  x'^\mu = \alpha x^\mu \\
  D = -i x^\mu \partial_\mu 
  \]

- **Rotation**:
  
  \[
  x'^\mu = M^\mu_\nu x^\nu \\
  L_{\mu\nu} = i(x^\mu \partial_\nu - x^\nu \partial_\mu) 
  \]

- **Special Conformal Transformation**:
  
  \[
  x'^\mu = \frac{x^\mu - (x.x)b^\mu}{1 - 2(b.x) + (b.b)(x.x)} \\
  K_\mu = -i(2x_\mu x^\nu \partial_\nu - (x.x) \partial_\mu) 
  \]

The total number of generators, it is clear from the indices on the generators, are $(d+2)(d+1)/2$. Further, if we define:

\[
J_{\mu\nu} = L_{\mu\nu}; J_{-1,\mu} = \frac{1}{2}(P_\mu - K_\mu); J_{-10} = D; J_{0,\mu} = \frac{1}{2}(P_\mu + K_\mu) \quad (1.23)
\]

where $m, n = -1, 0, ... d$

then these satisfy the usual $SO$ group commutation relation:

\[
[J_{mn}, J_{rs}] = i(\eta_{ms}J_{nr} + \eta_{mr}J_{ns} - \eta_{mr}J_{ns} - \eta_{ns}J_{mr}) \quad (1.24)
\]

Hence, The Conformal group hence corresponds to the $SO$ algebra in d dimensions where $d \geq 3$. 
1.4 Conformal Group in d=2

Earlier, we had restricted our discussion of the Conformal Group to \( d \geq 3 \). Now let us look at the Conformal Group in \( d = 2 \). One may ask why we could not generalize the treatment in the previous question to \( d = 2 \). It turns out, as we will notice in a moment, that the conformal group is infinitely large in \( d = 2 \) and our earlier treatment does not make that obvious. The globally defined transformations in the \( d \geq 3 \) case, it should be kept in mind, are the same as in the \( d = 2 \) case.

The condition given in (1.6) for invariance under conformal transformations leads in \( d = 2 \) to:

\[
\begin{align*}
\partial_0 \epsilon_0 &= \partial_1 \epsilon_1 \quad (1.25) \\
\partial_0 \epsilon_1 &= -\partial_1 \epsilon_0 \quad (1.26)
\end{align*}
\]

Where the signature of the metric is \((-1,1)\). This we recall are nothing but the Cauchy-Riemann equations. As a result, we know that functions of the form \( f(z) = z + \epsilon(z) \), that is, holomorphic functions, satisfy the given conditions. To make the treatment easier, we introduce complex variables. To do it we make the following definitions:

\[
\begin{align*}
z &= x^0 + \iota x^1; \epsilon = \epsilon^0 + \iota \epsilon^1; \partial_z = \frac{1}{2} (\partial_0 - \iota \partial_1) ; \\
\bar{z} &= x^0 - \iota x^1; \bar{\epsilon} = \epsilon^0 - \iota \epsilon^1; \partial_{\bar{z}} = \frac{1}{2} (\partial_0 + \iota \partial_1)
\end{align*}
\]

Equations (1.25) and (1.26) we recall represents nothing but the Cauchy-Riemann equations. As a result, we know that functions of the form \( f(z) = z + \epsilon(z) \), that is, holomorphic functions, satisfy the given conditions. Hence, in 2\( d \), the infinitesimal conformal transformation is:

\[
z \rightarrow f(z)
\]

Where,

\[
f(z) = z + \epsilon(z)
\]

Next, we expand \( \epsilon(z) \) about \( z = 0 \) which leads to an infinite Laurent series:

\[
\bar{z}' = \bar{z} + \bar{\epsilon}(\bar{z}) = \bar{z} + \sum_n \bar{\epsilon}_n(-\bar{z}^{n+1})
\]
\[ z' = z + \epsilon(z) = z + \sum_n \epsilon_n(-z^{n+1}) \]

Here \( \epsilon(n) \) and \( \overline{\epsilon}(n) \) are constants.

The generators then are given by:

\[ l_n = -z^{n+1}\partial_z \quad \text{and} \quad \overline{l}_n = -\overline{z}^{n+1}\partial_{\overline{z}} \]

which follow the following commutation relations:

\[ [l_m, l_n] = (m - n)l_{m+n} \]
\[ [\overline{l}_m, \overline{l}_n] = (m - n)\overline{l}_{m+n} \]
\[ [l_m, \overline{l}_n] = 0 \]

This is called the Witt algebra. Clearly, there is an infinite number of linearly independent generators which leads us to the important fact that: In 2d, the conformal group is infinite dimensional. The globally defined transformations on the Riemann sphere \( S^2 = C \cup \infty \) are ones that are well defined at \( z = 0 \) and \( z = \infty \). From the form of the generators, we see here that these are generated by \( l_0, l_{-1}, l_1 \). In order to see what these transformations represent, we notice the following:

- \( l_{-1} = -\partial_z \) and hence \( l_{-1} \) generates translations.
- \( l_0 + \overline{l}_0 = -r\partial_r \), \( \iota(l_0 - \overline{l}_0) = -\partial_\phi \)
  \[ l_0 = -\frac{1}{2}r\partial_r + \frac{i}{2}\partial_\phi \], \( \overline{l}_0 = -\frac{1}{2}r\partial_r - \frac{i}{2}\partial_\phi \)
  where we used \( z = re^{i\phi} \)
- It is clear then that \( l_0 + \overline{l}_0 = -r\partial_r \) generates dilations and \( \iota(l_0 - \overline{l}_0) \) generates rotations.
- The operator \( l_1 = -z^2\partial_z \) generates infinitesimal transformations of the form:

\[ z \rightarrow \frac{z}{cz+1} \]

Where \( c \) is some constant. Comparing this to the general form of Special Conformal Transformations, we see that this is a Special Conformal Transformation, with the \((x,b)\) term as given in generic expression equal to zero.
We define a central extension of these algebras called the Virasoro algebra, we impose that:

\[ [\tilde{x}, \tilde{y}]_{\tilde{g}} = [x, y]_{g} + cp(x, y) \] (1.27)

\[ \Rightarrow [L_{m}, L_{n}] = (m - n)L_{m+n} + cp(m, n) \] (1.28)

where \( p(m, n) \) is antisymmetric in \( m, n \) and \( c \) is called the central charge.

### 1.5 Primary Fields

Consider the coordinate transformation, \( z \rightarrow f(z) \). Then fields that transform as:

\[ \Phi(z, \bar{z}) \rightarrow \Phi'(z, \bar{z}) = (\frac{\partial f}{\partial z})^{h} (\frac{\partial f}{\partial \bar{z}}) \Phi(f(z), f(\bar{z})) \] (1.29)

are called primary fields with conformal dimensions \( (h, \bar{h}) \). If the equation above holds only for global transformations, then the corresponding field is called a quasi-primary field. Note that all primary fields are quasi primary fields but the converse is not true.

#### Infinitesimal Conformal Transformations of Primary Fields

To investigate how primary fields transform under infinitesimal conformal transformations, consider a map \( f(z) = z + \epsilon(z) \) with \( \epsilon(z) << 1 \) and compute the following up to first order in \( \epsilon(z) \):

\[ (\frac{\partial f}{\partial z})^{h} = 1 + h\partial_{z}\epsilon(z) + O(\epsilon^{2}) \]

\[ \Phi(z + \epsilon(z), \bar{z}) = \Phi(z) + \epsilon(z)\partial_{z}\Phi(z, \bar{z}) + O(\epsilon^{2}) \]

Using these two expressions in the definition of a primary field, we get the following:

\[ \Phi(z, \bar{z}) \rightarrow \Phi(z, \bar{z}) + (h\partial_{z}\epsilon + \epsilon\partial_{z} + \bar{h}\partial_{\bar{z}}\epsilon + \epsilon\partial_{\bar{z}})\Phi(z, \bar{z}) \]

Hence, the under an infinitesimal conformal transformation, a primary field transforms in the following manner:

\[ \delta_{\epsilon, \bar{\epsilon}}\Phi(z, \bar{z}) = (h\partial_{z}\epsilon + \epsilon\partial_{z} + \bar{h}\partial_{\bar{z}}\epsilon + \epsilon\partial_{\bar{z}})\Phi(z, \bar{z}) \] (1.30)
1.6 The Energy Momentum Tensor

Noether’s theorem tells us that for every continuous symmetry in a Lagrangian, there is a conserved current, i.e. (in flat space) \( \partial^\mu j_\mu = 0 \). For transformations of the type \( x^\mu \rightarrow x^\mu + \epsilon^\mu(x) \) the conserved current is of the form:

\[
j_\mu = T_{\mu \nu} \epsilon^\nu
\]

Where \( T_{\mu \nu} \) is a symmetric tensor called the Energy Momentum Tensor. For the special case when \( \epsilon^\mu \) is constant, the current conservation condition implies that

\[
\partial^\mu T_{\mu \nu} = 0 \quad (1.31)
\]

For the general case when it is not a constant, we have the following:

\[
0 = \partial^\mu j_\mu = (\partial^\mu T_{\mu \nu}) \epsilon^\nu + T_{\mu \nu} (\partial^\mu \epsilon^\nu) = 0 + \frac{1}{2} T_{\mu \nu} (\partial^\mu \epsilon^\nu + \partial^\nu \epsilon^\mu)
\]

\[
= \frac{1}{2} T_{\mu \nu} \eta^\mu \nu (\partial. \epsilon) = \frac{1}{2} \partial_\mu (\partial. \epsilon)
\]

where we used (1.6) and the fact that \( T_{\mu \nu} \) is a symmetric tensor. For the equation to hold for any arbitrary \( \epsilon \) we conclude that:

The Energy Momentum Tensor is traceless in Conformal Field Theories. \( (T^\mu_\mu = 0) \)

This fact is often used to check conformal invariance in lagrangians.

The EM tensor in two euclidean dimensions

Using the coordinate change: \( x^0 = \frac{1}{2} (z + \bar{z}) \) and \( x^1 = \frac{1}{2i} (z - \bar{z}) \) we can write the components of the Energy Momentum Tensor as follows:

\[
T_{zz} = \frac{1}{4} (T_{00} - 2i T_{10} - T_{11}) \; , \; T_{z \bar{z}} = T_{\bar{z} z} = \frac{1}{4} (T_{00} + T_{11}) = \frac{1}{4} T^\mu_\mu = 0 
\]

\[
T_{\bar{z} \bar{z}} = \frac{1}{4} (T_{00} + 2i T_{10} - T_{11})
\]

Where we used \( \eta_{\mu \nu} = diag(1,1) \) and the fact that \( T^\mu_\mu = 0 \)

Using the tracelessness of \( T_{\mu \nu} \) we get

\[
T_{zz} = \frac{1}{2} (T_{00} - i T_{10}) \; , \; T_{z \bar{z}} = \frac{1}{2} (T_{00} + i T_{10}) \quad (1.32)
\]

Using here the condition (1.31), we find that:

\[
\partial_0 T_{00} + \partial_1 T_{10} = 0 \; , \; \partial_0 T_{01} + \partial_1 T_{11} = 0 \quad (1.33)
\]
1.7. RADIAL QUANTIZATION

Using this equation along with (1.33) and the tracelessness of the $T_{\mu\nu}$ we get that $\partial_z T_{\overline{z}\overline{z}}$ and $\partial_{\overline{z}} T_{zz}$.

Hence, the two non zero components of the energy-momentum tensor are

$$ T_{zz}(z, \overline{z}) =: T(z), T_{\overline{z}\overline{z}}(z, \overline{z}) =: \overline{T}(\overline{z}) $$

1.7 Radial Quantization

We now learn how to quantize our fields which is one of the usual objectives in any field theory. For now, we will restrict ourselves to two dimensions. We define our time coordinate to be $x^0$ and our space coordinate to be $x^1$. We then make the following change of variables:

$$ z = e^w = e^{x^0} \cdot e^{i x^1} \quad (1.34) $$

We see that time translations correspond to complex dilations and space translations correspond to rotations.

1.7.1 Asymptotic States

Consider a field $\Phi(z, \overline{z})$ with conformal dimensions $(h, \overline{h})$ for which we can do a Laurent expansion around $z_0 = \overline{z}_0 = 0$ in the following manner:

$$ \Phi(z, \overline{z}) = \sum_{n, \overline{m} \in \mathbb{Z}} z^{-n-h} \overline{z}^{-\overline{m} - \overline{h}} \Phi_{n, \overline{m}} \quad (1.35) $$

Now we move to quantizing this field. This is achieved by promoting the modes $\Phi_{n, \overline{m}}$ in the Laurent expansion (1.35) to operators.

Equation (1.34) tells us that the infinite past $x^0 = -\infty$ is mapped to $z = \overline{z} = 0$ which motivates the following definition of an asymptotic in state $|\Phi\rangle$ to be in the following form:

$$ |\Phi\rangle = \lim_{z, \overline{z} \to 0} \Phi(z, \overline{z}) |0\rangle \quad (1.36) $$

In order for this to be well defined at $z = 0$ we must impose that

$$ \Phi_{n, \overline{m}} = |0\rangle \text{ for } n > -h, \overline{m} > -\overline{h} $$
Using this result and the expression for Laurent expansion (1.35) we can simplify (1.36) to the following:

\[
|\Phi\rangle = \lim_{z,\bar{z} \to 0} \Phi(z, \bar{z}) |0\rangle = \Phi_{-h, -\bar{h}} |0\rangle \tag{1.37}
\]

1.7.2 Hermitian Conjugation

If we define \(x^0\) to be not the usual time but the euclidean time i.e \(x^0 = it\) then we see that \(x^0 \to -x^0\) under complex conjugation. Our complex coordinate \(z = exp(x^0+i\mathcal{A})\) under hermitian conjugation transforms to \(\bar{z}\).

Using our knowledge of how primary fields transform, we define the hermitian conjugate of a field by:

\[
\Phi^\dagger = z^{-2h} \bar{z}^{-2\mathcal{A}} \Phi(1/z, 1/\bar{z}) \tag{1.38}
\]

Performing a laurent’s expansion of this field then gives:

\[
\Phi^\dagger(z, \bar{z}) = z^{-2h} \bar{z}^{-2\mathcal{A}} \sum_{n,m\in\mathbb{Z}} z^n z^m + \bar{z}^n \bar{z}^m \Phi_{n,m} = \sum_{n,m\in\mathbb{Z}} z^{-n-h} \bar{z}^{-m-\mathcal{A}} \Phi_{n,m} \tag{1.39}
\]

Comparing this expression with the hermitian conjugate of equation (1.35) we find that:

\[
(\Phi_{n,m})^\dagger = \Phi_{-n,-m} \tag{1.40}
\]

Now we proceed to define an expression for an assymptotic out state. This is, naturally, done by using the hermitian conjugate field:

\[
\langle \Phi | = \lim_{z,\bar{z} \to 0} \langle 0 | \Phi^\dagger(z, \bar{z}) = \lim_{w,\bar{w} \to \infty} w^{2h} \bar{w}^{2\mathcal{A}} \langle 0 | \Phi(w, \bar{w})
\]

where we have used equation (1.38) and the definitions \(\bar{z} = w^{-1}\) and \(z = \bar{w}^{-1}\)

Following on the same reasoning as for the case of the in-state, we require this state to be well defined and hence impose that

\[
\langle 0 | \Phi_{n,m} = 0 \text{ for } n < h, m < \bar{h}
\]

Recalling the Laurent expansion of the field, this then leads us to the following:

\[
\langle \Phi | = \lim_{w,\bar{w} \to \infty} w^{2h} \bar{w}^{2\mathcal{A}} \langle 0 | \Phi(w, \bar{w}) = \langle 0 | \Phi_{+h, +\bar{h}} \tag{1.41}
\]
1.8 The Operator Product Expansion

This section aims at looking at the energy-momentum tensor in more detail and to introduce the operator formalism for two-dimensional conformal field theories.

Conserved Charges

To start with, we recall that corresponding to every conformal symmetry, there is a conserved current given by \( j_\mu = T_{\mu\nu}\epsilon^\nu \). There exists a corresponding conserved charge which is given by

\[
Q = \int dx^1 j_0, \text{ at } x^0 = \text{constant}
\]

We know from field theory that this conserved charge is the generator of symmetry transformations for an operator \( A \) which can be written as follows:

\[
\delta A = [Q, A] \tag{1.42}
\]

where this commutator is at equal times. From (1.34) we see that constant \( x^0 \) implied \(-z-\)constant. Hence, the integral over \( dx^1 \) becomes a contour integral. The obvious generalization of the conserved charge equation to complex coordinates is:

\[
Q = \frac{1}{2\pi i} \oint_C (dz T(z)\epsilon(z) + dz\bar{T}(\bar{z})\bar{\epsilon}(\bar{z})) \tag{1.43}
\]

Using (1.43), the infinitesimal transformation of the \( \Phi \) field \( \Phi(z, \bar{z}) \) is given by:

\[
\delta_{\epsilon, \epsilon} \Phi(w, \bar{w}) = \frac{1}{2\pi i} \oint_C dz[T(z)\epsilon(z), \Phi(w, \bar{w})] + \frac{1}{2\pi i} \oint_C d\bar{z}[T(\bar{z})\bar{\epsilon}(\bar{z}), \Phi(w, \bar{w})] \tag{1.44}
\]

1.8.1 Radial Ordering

If we look closely at the above equation, there is some ambiguity as to whether or not \( w \) and \( \bar{w} \) are inside the contour \( C \) or outside it. We know from quantum field theory that correlation functions are only defined as time ordered products. We also know from our coordinate change (1.34) that in CFTs time ordering becomes radial ordering and hence the product \( A(z)B(w) \) is defined only when \(|z| > |w|\). Subsequently, the radial ordering of two operators is defined as:
\[ R(A(z)B(w)) = A(z)B(w), \text{ for } |z| > |w| \]  
\[ R(A(z)B(w)) = B(w)A(z), \text{ for } |w| > |z| \]  

Therefore, an expression of the form takes the form:

\[ \oint dz[A(z), B(w)] = \oint |z|>|w| dzA(z)B(w) - \oint |z|<|w| dzB(w)A(z) = \oint_{C(w)} dzR(A(z)B(w)) \]  

Using (1.47) we can now express (1.44) as follows:

\[ \delta_{\epsilon,\tilde{\epsilon}}\Phi(w, \bar{w}) = \frac{1}{2\pi i} \oint_{C(w)} dz \epsilon(z) R(T(z)\Phi(w, \bar{w})) + \frac{1}{2\pi i} \oint_{C(\bar{w})} d\bar{z} \tilde{\epsilon}(\bar{z}) R(\bar{T}(\bar{z})\Phi(w, \bar{w})) \]  

Where since \( \epsilon(z) \) and \( \tilde{\epsilon}(\bar{z}) \) are not operators, we can pull them out of the radially ordered expression. Next, comparing this with (1.44) we deduce that:

\[ R(T(z)\Phi(w, \bar{w})) = \frac{h}{(z-w)^2} \Phi(w, \bar{w}) + \frac{1}{z-w} \partial_w \Phi(w, \bar{w}) + ... \]  

Where we have used the following identities:

\[ h(\partial_w \epsilon(w))\Phi(w, \bar{w}) = \frac{1}{2\pi i} \oint_{C(w)} dz h \frac{\epsilon(z)}{(z-w)^2} \Phi(w, \bar{w}) \]

\[ \epsilon(w)(\partial_w \Phi(w, \bar{w})) = \frac{1}{2\pi i} \oint_{C(w)} dz \frac{\epsilon(z)}{z-w} \partial_w \Phi(w, \bar{w}) \]

The proof of these identities involves use of the residue theorem and is skipped here in favor of brevity.

An expression such as (1.49) is what is called an Operator Product Expansion. This also allows us to write an alternate definition for a primary field: A field \( \Phi(z, \bar{z}) \) whose operator product expansion with the energy momentum tensor takes the following form:

\[ T(z)\Phi(w, \bar{w}) = \frac{h}{(z-w)^2} \Phi(w, \bar{w}) + \frac{1}{(z-w)} \partial_w \Phi(w, \bar{w}) + ... \]  
\[ \bar{T}(\bar{z})\Phi(w, \bar{w}) = \frac{\bar{h}}{(\bar{z}-\bar{w})^2} \Phi(w, \bar{w}) + \frac{1}{(\bar{z}-\bar{w})} \partial_w \Phi(w, \bar{w}) + ... \]  

is called a primary field with conformal dimensions \( (h, \bar{h}) \).
1.8.2 Operator Product Expansion of the Energy Momentum Tensor

Here, we state the OPE of the energy momentum tensor and then attempt to justify it. The OPE is given by:

\[ T(z)T(w) = \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \ldots \] (1.52)

A similar equation holds for \( \overline{T}(\bar{z}) \)

To prove this, we do a Laurent expansion:

\[ T(z) = \sum_{n \in \mathbb{Z}} z^{-n-2} L_n, \text{ where } L_n = \frac{1}{2\pi i} \oint dz z^{n+1} T(z) \] (1.53)

Using the equation for conserved charge (1.43) and a conformal transformation \( \epsilon(z) = -(\epsilon_n z^{n+1}) \) we get:

\[ Q_n = \oint \frac{dz}{2\pi i} T(z)(-\epsilon_n z^{n+1}) = \epsilon_n \sum_{m \in \mathbb{Z}} \oint \frac{dz}{2\pi i} L_m z^{n-m-1} = -\epsilon_n L_n \] (1.54)

Since \( Q_n = \epsilon_n L_n \) the Laurent modes are proportional to the generators of infinitesimal conformal transformations. Consequently, they must satisfy the Virasoro algebra which will be a check on our OPE. Using the expression for \( L_m \) given above, it can be shown that:

\[ [L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n} \]

This commutation relation of the Laurent modes matches the commutation relation for the Virasoro algebra given earlier with \( p = \frac{1}{12}(m^3 - m)\delta_{m,-n} \) and therefore, we conclude that (45) is the correct expansion.

1.9 Operator Algebra of Chiral Quasi Primary Fields

Our objective now is to consider n-point correlation functions and try and develop a systematic approach to evaluate them exactly where possible.
The Conformal Ward Identity

For primary fields $\Phi_i$, we consider the following:

$$\left\langle \oint \frac{dz}{2\pi i} \epsilon(z) T(z) \Phi_1(w_1, \overline{w}_1) \Phi_2(w_2, \overline{w}_2) \ldots \Phi_n(w_n, \overline{w}_n) \right\rangle \quad (1.55)$$

$$= \sum_{i=1}^N \left( \Phi_1(w_1, \overline{w}_1) \ldots \left( \oint_{C_i} \frac{dz}{2\pi i} \epsilon(z) T(z) \Phi_i(w_i, \overline{w}_i) \right) \ldots \Phi_n(w_n, \overline{w}_n) \right)$$

$$= \sum_{i=1}^N \left( h_i \partial_{w_i} \epsilon(w_i) + \epsilon(w_i) \partial_{w_i} \Phi_i(w_i, \overline{w}_i) \right) \ldots \Phi_n(w_n, \overline{w}_n)$$

where in the last step, we have used (1.48).

Now employing the identities stated earlier, we write:

$$0 = \left\langle \oint \frac{dz}{2\pi i} \epsilon(z) T(z) \Phi_1(w_1, \overline{w}_1) \Phi_2(w_2, \overline{w}_2) \ldots \Phi_n(w_n, \overline{w}_n) \right\rangle$$

$$- \sum_{i=1}^N \left( h_i \partial_{w_i} \epsilon(w_i) + \epsilon(w_i) \partial_{w_i} \Phi_i(w_i, \overline{w}_i) \right) \ldots \Phi_n(w_n, \overline{w}_n)$$

Since this relation must hold for all $\epsilon(z) = -z^{n+1}$, the following relation holds identically and is referred to as the Ward identity.

$$\left\langle \oint \frac{dz}{2\pi i} \epsilon(z) T(z) \Phi_1(w_1, \overline{w}_1) \Phi_2(w_2, \overline{w}_2) \ldots \Phi_n(w_n, \overline{w}_n) \right\rangle \quad (1.56)$$

$$= \sum_{i=1}^N \left( \frac{h_i}{(z-w_i)^2} + \frac{1}{z-w_i} \partial_{w_i} \right) \Phi_1(w_1, \overline{w}_1) \Phi_2(w_2, \overline{w}_2) \ldots \Phi_n(w_n, \overline{w}_n)$$

### 1.9.1 Two and Three point functions

Let us assume the following general form for the two-point function of two chiral, holomorphic quasi-primary fields.

$$\left\langle \Phi_i(z) \Phi_j(w) \right\rangle = g(z, w)$$

Our task now is to determine a form for $g(z, w)$ We know that this function must be invariant under conformal transformations. Invariance under translations implies that $g(z, w) = g(z - w)$ Secondly, invariance under dilations implies that:

$$\left\langle \Phi_1(z) \Phi_2(w) \right\rangle \rightarrow \left\langle \lambda^{h_1} \Phi_1(\lambda z) \lambda^{h_2} \Phi_2(\lambda w) \right\rangle = \lambda^{h_1 + h_2} g(\lambda(z - w))$$
For this to equal $g(z - w)$ it must have the following form:

$$\langle \Phi_i(z) \Phi_j(w) \rangle = \frac{d_{ij}}{(z - w)^{h_1 + h_2}} \quad (1.57)$$

Finally, invariance under special conformal transformations: $f(z) = \frac{-1}{z}$ leads to

$$\langle \Phi_1(z) \Phi_2(w) \rangle \rightarrow \langle \frac{1}{z^{h_1}} \frac{1}{w^{h_2}} \Phi_1(-\frac{1}{z}) \Phi_2(-\frac{1}{w}) \rangle$$

$$\rightarrow \frac{1}{z^{h_1}w^{h_2}} \frac{d_{12}}{(-\frac{1}{z} + \frac{1}{w})^{h_1 + h_2}}$$

For this to equal $\frac{d_{ij}}{(z - w)^{h_1 + h_2}}$, it must be the case that $h_1 = h_2$ and hence the two point function is given by

$$\langle \Phi_i(z) \Phi_j(w) \rangle = \frac{d_{ij} \delta_{h_i, h_j}}{(z - w)^{2h_i}} \quad (1.58)$$

Repeating these same considerations for three point functions we arrive at the following result:

$$\langle \Phi_1(z_1) \Phi_2(z_2) \Phi_3(z_3) \rangle = \frac{C_{123}}{z_1^{h_1 + h_2 + h_3} z_2^{h_2 + h_3} z_3^{h_3 + h_1} (z_1 + h_3 - h_2)} \quad (1.59)$$

Notice that the denominator in (1.58) should be invariant under multiplication by $\exp^{2\pi i}$. This leads us to the fact that the conformal dimension of a chiral, quasi primary field must be either an integer or a half integer.

It can be shown by using these results that the general expression for the OPE between two chiral, quasi primary fields is given by:

$$\Phi_i(z) \Phi_j(w) = \sum_{k, n \geq 0} C_{ijk}^k \frac{a_{ijk}^n}{n!} \frac{\partial^n \Phi_k(w)}{(z - w)^{h_i + h_j - h_k - n}} \quad (1.60)$$

where

$$a_{ijk}^n = \binom{2h_k + n - 1}{n}^{-1} \binom{h_k + h_i - h_j + n - 1}{n}$$

$$C_{ijk} = C_{ij}^l d_{lk}$$
1.9.2 General expression for the commutation relations

In this subsection, we will attempt to outline the result for the commutation relation between two chiral, quasi primary fields. The detailed proof requires tedious computations and will not be presented here.

Recall that the Laurent expansion of chiral fields $\Phi_i(z)$ is given by:

$$\Phi_i(z) = \sum_m \Phi_{(i)m} z^{-m-h_i}$$

where $h_i$ is the conformal dimension of a chiral, quasi primary field $\Phi_i$ which as proved earlier must be either an integer or a half integer. As a result, for this expression to be consistent with the generic expression for Laurent expansions, $m$ must also be an integer or a half integer, that is:

$$m \in \mathbb{Z} \text{ or } m \in \mathbb{Z} + \frac{1}{2}$$

then, just like we write expressions for the coefficients in a generic Laurent expansion, we express the Laurent modes, $\Phi_{(i)m}$’s in the expansion above as contour integrals over $\Phi_i(z)$. We then evaluate the commutator $[\Phi_{(i)m}, \Phi_{(j)n}]$ which eventually simplifies to:

$$[\Phi_{(i)m}, \Phi_{(j)n}] = \sum_k C_{ijk}^k p_{ijk}(m, n)\Phi_{(k)m+n} + d_{ij} \delta_{m,-n} \left( \frac{m+h_i-1}{2h_i-1} \right)$$

(1.61)

where

$$p_{ijk}(m, n) = \sum_{(r,s)\in\mathbb{Z}^+ \cap \{r+s=h_i+h_j-h_k-1\}} C_{r,s}^{ijk} \left( \frac{-m+h_i-1}{r} \right) \left( \frac{-n+h_j-1}{s} \right)$$

$$C_{r,s}^{ijk} = (-1)^r \frac{(2h_k-1)!}{[h_i+h_j+h_k-2]!} \prod_{t=0}^{s-1} (2h_i - 2 - r - t) \prod_{\mu=0}^{r-1} (2h_j - 2 - s - \mu)$$

1.10 Normal Ordered Products

Since $\Phi'_i$s are operators, we need to give a prescription for how their products act. This is where the notion of normal ordering comes in. Just to recall, in QFT’s, normal ordering implies ”creation operators to the left.”
1.10. NORMAL ORDERED PRODUCTS

Normal Ordering Prescription

Recall that:

$$\Phi_{n,m} = |0\rangle \text{ for } n > -h, m > -\bar{h}$$

From this it is apparent that we can interpret $$\Phi_{n,m}$$ with $$n > -h$$ or $$m > -\bar{h}$$ as annihilation operators. As a result, we conclude that:

- $$\Phi_n$$ with $$n > -h$$ are annihilation operators
- $$\Phi_n$$ with $$n \leq -h$$ are creation operators

1.10.1 Normal Ordered Products and OPEs

Remember that the normal ordering prescription requires all creation operators to be put the left. In this subsection, we will present a general expression for the OPE of two chiral, quasi primary fields using normal ordered products. Let us first state the result and then verify it:

$$\Phi(z)\chi(w) = \text{sing.} + \sum_{n=0}^{\infty} \frac{(z-w)^n}{n!}N(\chi \partial^n \Phi)(w) \quad (1.62)$$

where N denotes normal ordering and "sing." denotes terms with powers of $$(z-w)$$ in the denominator. The terms grouped together in "sing." comprise the irregular part of the expansion.

To verify this, let us begin by multiplying $$\frac{1}{2\pi i} \oint dz (z-w)^{-1}$$ on both sides. This picks out the $$n = 0$$ term while the other terms give zeros. Therefore:

$$N(\chi \Phi)(w) = \oint_{C(w)} \frac{dz}{2\pi i} \frac{\Phi(z)\chi(w)}{(z-w)} \quad (1.63)$$

We can also perform a Laurent expansion of $$N(\chi \Phi)$$ in the usual way:

$$N(\chi \Phi)(w) = \sum_{n \in \mathbb{Z}} w^{-n-h^\phi-h^\chi} N(\chi \Phi)_n$$

$$\implies$$

$$N(\chi \Phi)_n = \oint_{C(0)} \frac{dw}{2\pi i} w^{n+h^\phi+h^\chi-1} N(\chi \Phi)(w)$$
where $C(0)$ indicates a contour integral about the origin.

\[
N(\chi \Phi)_n = \oint_{C(0)} \frac{dw}{2\pi i} w^{n+h^\phi+h^\chi - 1} \oint_{C(w)} \frac{dz}{2\pi i} \frac{\Phi(z)\chi(w)}{(z-w)}
\]

Evaluating these integrals we get the following result:

\[
N(\chi \Phi)_n = \sum_{k> -h^\phi} \chi_{n-k} \Phi_k + \sum_{k \leq -h^\phi} \Phi_k \chi_{n-k}
\]  

(1.64)

We see, here, that $\Phi_k$ in the first term are annihilation operators and are to the right whereas in the second term, they are creation operators and are to the left. The expression, therefore, really is normal ordered. We have thus verified that the expression (1.62) holds for $n=0$. The proof that it is true in general is more involved and is beyond the scope of the discussion here. We will therefore skip the general proof.

By a similar computation, we can also derive the following important results:

\[
N(\chi \partial \Phi)_n = \sum_{k> -h^\phi - 1} (-h^\phi - k) \chi_{n-k} \Phi_k + \sum_{k \leq -h^\phi - 1} (-h^\phi - k) \Phi_k \chi_{n-k}
\]  

(1.65)

\[
N(\partial \chi \Phi)_n = \sum_{k> -h^\phi} (-h^\chi - n + k) \chi_{n-k} \Phi_k + \sum_{k \leq -h^\phi} (-h^\chi - n + k) \Phi_k \chi_{n-k}
\]  

(1.66)

This completes our discussion of the essential elements of conformal field theories in flat space. Details of the CFT hilbert space and examples of CFT’s discussed in [1] have been skipped to keep the discussion as concise as possible. The things discussed in this chapter are essentially everything that is required to understand the things that will come up throughout the rest of this thesis.
Chapter 2

Anti de Sitter space

Having looked at CFT’s, we now look at the basics that underlie the other half of the AdS/CFT duality. The discussion here borrows from [9]

2.1 Introduction

An Anti de Sitter space is a solution to the vacuum Einstein Field equations with a cosmological constant. It is a Lorentizan manifold, that is an n dimensional Riemannian manifold whose metric is not positive definite and has the signature (-1,+1,+1,+1,+1,...,+1). The scalar curvature of an AdS space is by definition a negative constant. The proof is as follows:

Recall that the vacuum Einstein equation with a cosmological constant is given by:

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{2} \Lambda g_{\mu\nu} \]  

(2.1)

Taking the trace on both sides:

\[ g^{\mu\nu} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} g^{\mu\nu} R = \frac{1}{2} \Lambda g_{\mu\nu} g^{\mu\nu} \]

\[ \implies R_\mu^\mu - \frac{1}{2} n R = \frac{1}{2} n \Lambda \]

\[ \implies R = -\frac{n}{n-2} \Lambda \]

where R denotes the Ricci scalar and n is the dimension of the space. We can see that we have a spacetime that has a negative curvature in the absence of matter. Next, we use the expression for R in equation (2.1) to get the following:
\[ R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} \left( -\Lambda \frac{n}{n-2} \right) + \Lambda g_{\mu\nu} = \frac{-\Lambda}{n-2} g_{\mu\nu} \]  

(2.2)

Hence we have a case where the Ricci tensor is proportional to the metric tensor. Moreover, an AdS space is a maximally symmetric manifold. In non mathematical terms: it is homogenous (invariant under translations) and isotropic (invariant under rotations)

Mathematically, it can be shown that it translates into the following condition:

\[ R_{\mu\nu\rho\sigma} = \frac{R}{n(n-1)} (g_{\mu\rho} g_{\nu\sigma} - g_{\nu\rho} g_{\mu\sigma}) \]  

(2.3)

where \( R_{\mu\nu\rho\sigma} \) denotes the Riemann tensor.

### 2.2 Geometry of AdS space

Embedded in \( n + 1 \) dimensions, an \( n \) dimensional AdS space is the set of points for which \( y^2 = l^2 \) where the embedding is Lobachevski like, that is where:

\[ y^2 = y^\mu y_\mu = (y^0)^2 + (y^{n+1})^2 - \sum_{i=1}^{n} (y^i)^2 \]

We can prove this by checking that this space satisfies the above conditions for a manifold to be an AdS space Let us proceed with the proof as follows:

Consider an \( n + 2 \) dimensional space. On it consider the set of points that satisfy:

\[ y^2 = l^2 \]

Let us define our coordinates as follows:

\[ y^0 = l \frac{1+x_n x^n}{1-x_n x^n} \]

\[ y^i = l \frac{2x^i}{1-x_n x^n} \]

\[ \implies \]

\[ y^2 = \frac{l^2 (1+(x_n x^n))^2}{(1-x_n x^n)^2} + \frac{4l^2 (x^{n+1})^2}{(1-x_n x^n)^2} - \frac{4l^2 \sum_{i=1}^{n} (x^i)^2}{(1-x_n x^n)^2} \]
2.2. GEOMETRY OF ADS SPACE

\[ y^2 = l^2 \frac{1 + (x_a x^a)^2 - 2(x_a x^a)}{(1 - x_a x^a)^2} = l^2 \]

Let us now compute the metric of this \((n+1)\) dimensional space.

\[ dy^0 = \frac{2l}{1 - x_a x^a} x_i dx^i + l \frac{1 + x_a x^a}{(1 - x_a x^a)^2} 2x_i dx^i = 4l \frac{(x_i dx^i)}{(1 - x.x)^2} \]  

\[ dy^i = \frac{2l}{(1 - x.x)^2} ((1 - x.x)\delta^i_k + 2x^i x_k) dx^k \]

Using this,

\[ ds^2 = (dy^0)^2 + (dy^{n+1})^2 - \sum_{i=1}^{n} (dy^i)^2 = \frac{-4l^2 dx^2}{(1 - x.x)^2} \]

where

\[ dx^2 = \sum_{i=1}^{n} (dx^i)^2 - (dx^{n+1})^2 \]

Therefore,

\[ g_{\mu\nu} = \frac{-4l^2}{(1 - x.x)^2} \eta_{\mu\nu} \]

Such a metric is called a conformally flat metric. Next, we compute the christoffels:

\[ \Gamma^\mu_{\nu\sigma} = \frac{1}{2} g^{\mu\alpha} (\partial_{\nu} g_{\sigma\alpha} + \partial_{\sigma} g_{\nu\alpha} - \partial_{\alpha} g_{\nu\sigma}) = \frac{1}{2} \left( \frac{4x_\nu}{(1 - x.x)} \delta^\mu_\sigma + \frac{4x_\sigma \delta^\mu_\nu}{(1 - x.x)} - \frac{4x_\mu \eta_{\nu\sigma}}{(1 - x.x)} \right) \]

Next, we use the chrifoffel symbols to evaluate the Riemann tensor. We get:

\[ R_{\mu\nu\sigma\alpha} = \frac{1}{l^2} (g_{\nu\alpha} g_{\mu\sigma} - g_{\nu\sigma} g_{\mu\alpha}) \]

The Ricci tensor is given by

\[ R_{\nu\alpha} = \frac{1}{l^2} (g_{\nu\alpha} \delta^\mu_\mu - g_{\nu\mu} \delta^\mu_\alpha) = \frac{1}{l^2} (g_{\nu\alpha}(n + 1) - g_{\nu\alpha}) = \frac{n}{l^2} g_{\nu\alpha} \]

The Ricci scalar is obtained by tracing the above equation and is given by:

\[ \frac{n(n+1)}{l^2} \]

Here \( n \) is the dimension of the space and so in our case \( n \) is actually \( n+1 \). Therefore, since
we see from eq(2.9) that the condition for maximal symmetricity is satisfied. Also by comparing eq(2.2) with (2.10) we see that

\[ \Lambda = \frac{-n(n-1)}{l^2} \]

We have thus verified that our \( n + 1 \) dimensional space is indeed an AdS space. Hence, an AdS space, in an embedded description, is a locus of points that satisfy \( y^2 = l^2 \) where \( y^2 = (y^0)^2 + (y^{n+1})^2 - \sum_{i=1}^{n}(y^i)^2 \). Moreover, the isometry group, clearly is SO \( (d,2) \) where \( d \) is the dimension of the space. This corresponds to the isometry group for CFT’s in \( d \geq 3 \). This was one of the main motivations for the AdS/CFT correspondence conjecture which will be discussed in detail in the later chapters.

### 2.3 Boundary of \( AdS_{n+1} \) space

The boundary of an \( AdS_{n+1} \) is the set of points that satisfy the condition that \( y^\mu \to \infty, y^\mu \in AdS_{n+1} \). Let us define \( y^\mu = R\tilde{y}^\mu \), \( R \to \infty \). From the condition \( y^2 = l^2 \) stated in the previous section, we have:

\[ R^2((\tilde{y}^0)^2 + (\tilde{y}^{n+1})^2 - \sum_{i=1}^{n}(\tilde{y}^i)^2) = l^2 \implies (\tilde{y}^0)^2 + (\tilde{y}^{n+1})^2 - \sum_{i=1}^{n}(\tilde{y}^i)^2 = 0 \]

where in the last step, we divide both sides by \( R^2 \) and take the limit \( R \to \infty \).

The relationship that

\[ (\tilde{y}^0)^2 + (\tilde{y}^{n+1})^2 = \sum_{i=1}^{n}(\tilde{y}^i)^2 \]

means that any equation satisfied by \( y^\mu \) is also satisfied by \( ty^\mu \) where \( t \) is any constant. This means we can scale our coordinates \( y^\mu \) such that we have the following relationship:

\[ (\tilde{y}^0)^2 + (\tilde{y}^{n+1})^2 = 1 = \sum_{i=1}^{n}(\tilde{y}^i)^2 \]

This shows that the boundary of \( AdS_{n+1} \) space is \( S^1 \times S^{n-1} \).
2.4 $AdS_{2+1}$ space

Here, we take a look at a specific example of an AdS space, $AdS_{2+1}$. It is easy to draw certain results for this particular choice of space and these results can then be generalized. To do so, we follow the Lobachevski embedding prescription and proceed as follows:

Define the locus of points

$$x^2 + y^2 - w^2 - z^2 = -l^2$$

(2.10)

Consider the following change of variables:

$$w = l \cosh(\mu) \sin(t)$$

and

$$z = l \cosh(\mu) \cos(t)$$

where

$$\mu \in [0, \infty), t = [0, 2\pi)$$

The metric is then given by

$$ds^2 = dx^2 + dy^2 - dw^2 - dx^2 = dx^2 + dy^2 - l^2 \sinh^2(\mu) d\mu^2 - l^2 \cosh^2(\mu) dt^2$$

(2.11)

Let us now make another change of variables, this time for $x$ and $y$:

$$x = l \sinh(\mu) \cos(\phi)$$

(2.12)

and

$$y = l \sinh(\mu) \sin(\phi)$$

(2.13)

where $\phi \in [0, 2\pi)$

This leads to

$$ds^2 = l^2 \sinh^2(\mu) d\mu^2 - l^2 \cosh^2(\mu) dt^2 + l^2 \cosh^2(\mu) d\mu^2 + l^2 \sinh^2(\mu) d\phi^2$$

$$= -l^2 \cosh^2(\mu) dt^2 + l^2 d\mu^2 + l^2 \sinh^2(\mu) d\phi^2$$

using $\cosh^2(x) - \sinh^2(x) = 1$
For higher dimensional AdS spaces this generalizes to

\[-l^2 \cosh^2 dt^2 + l^2 d\mu^2 + l^2 \sinh^2(\mu) d\Omega_{d-2}^2\] (2.14)

To get a convenient form of the metric, we define \(\sinh(\mu) = \frac{r}{l}\), and the metric takes the following form:

\[ds^2 = -(l^2 + r^2) dt^2 + \left(1 + \frac{r^2}{l^2}\right)^{-1} dr^2 + r^2 d\phi^2\] (2.15)

This completes our discussion of AdS spaces.
Chapter 3

D branes

This chapter aims to highlight some important concepts related to D branes: how they arise in string theory, how gauge fields arise on their world volumes and finally, what form of the action that describes them. The discussion in this chapter borrows heavily from [2].

3.1 How do D branes arise?

From the concepts of general relativity, we know that particles trace out paths that minimize proper length, they trace out world lines. The natural generalization of this to a string is obviously a world sheet wherewith proper area is minimized. Consider an infinitesimal parallelogram on the world sheet with sides $d\vec{v}_1 = \frac{\partial \vec{x}}{\partial \xi_1} d\xi_1$ and $d\vec{v}_2 = \frac{\partial \vec{x}}{\partial \xi_2} d\xi_2$

where the world sheet is parametrized by $\xi^1$ and $\xi^2$.

From the formula for the area of a parallelogram, the infinitesimal area has the following expression:

$$dA = |d\vec{v}_1||d\vec{v}_2||\sin \theta| = \sqrt{|d\vec{v}_1|^2|d\vec{v}_2|^2 - (d\vec{v}_1 \cdot d\vec{v}_2)^2} \cos^2 \theta$$  \hspace{1cm} (3.1)

Writing this in terms of dot products gives:

$$dA = \sqrt{(d\vec{v}_1 \cdot d\vec{v}_1)(d\vec{v}_2 \cdot d\vec{v}_2 - (d\vec{v}_1 \cdot d\vec{v}_2)^2)}$$  \hspace{1cm} (3.2)

Using the expressions for $d\vec{v}_1$ and $d\vec{v}_2$, the proper area functional is subsequently given by:
CHAPTER 3. D BRANES

\[ A = \int d\xi^1 d\xi^2 \sqrt{\left( \frac{\partial \vec{\tau}}{\partial \xi^1} \cdot \frac{\partial \vec{\tau}}{\partial \xi^2} \right) \left( \frac{\partial \vec{\tau}}{\partial \xi^2} \cdot \frac{\partial \vec{\tau}}{\partial \xi^1} \right) - \left( \frac{\partial \vec{\tau}}{\partial \xi^1} \cdot \frac{\partial \vec{\tau}}{\partial \xi^1} \right)^2} \quad (3.3) \]

Now instead of calling our parameters \( \xi^1 \) and \( \xi^2 \), to give them the world sheet a more physical outlook we give, we replace with \( \tau \) and \( \sigma \). These are related to the space like and time like directions on the world sheet. The detail is skipped here in favor of brevity.

Now what is left is to give this expression the dimensions of action. The proper area is therefore multiplied by \( -\frac{T_0}{c} \). Switching to the convention of denoting world sheet coordinates by \( X^\mu \) in string theory and denoting:

\[ \dot{X}^\mu = \frac{\partial X^\mu}{\partial \tau} \quad (3.4) \]

and

\[ X'^\mu = \frac{\partial X^\mu}{\partial \sigma} \quad (3.5) \]

\[ S = -\frac{T_0}{c} \int_{\tau}^{\tau'} d\tau \int_{\sigma_1}^{\sigma_1} d\sigma \sqrt{\dot{X}^\cdot (X')^2 - (\dot{X}^\cdot (X')^2} \quad (3.6) \]

There are two things that need to be mentioned here. Firstly, in (3.6) the order of the terms is switched inside the radical. It can be shown that this is to ensure that the term under the radical is always positive. Secondly, the minus sign is put in there to ensure that the lagrangian has the correct form in the non-relativistic limit. Moreover, it is an easy exercise to show that the term in the radical in (3.6) is equal to \(-g\) where \( g \) is the determinant of the induced metric on the world sheet of the string. We can therefore write our action in the following nice, compact form:

\[ S = -\frac{T_0}{c} \int d\tau d\sigma \sqrt{-g} \quad (3.7) \]

The lagrangian is just a scalar and the measure is manifestly lorentz invariant. The action therefore is lorentz invariant. This is what is called the Nambu-Goto action.

Our next step is to vary the action in the usual way to derive the Euler-Lagrange equations. Using the following notation:

\[ P^\tau = \frac{\partial L}{\partial \dot{X}^\mu} \quad (3.8) \]
and

\[ P^\sigma_\mu = \frac{\partial L}{\partial X^{\sigma\mu}} \]  

(3.9)

the variation of the action, after getting rid of terms which contribute total derivatives, is given by:

\[
\delta S = \int_{\tau_i}^{\tau_f} d\tau \delta X^\mu (P^\sigma_\mu )^0 - \int_{\tau_i}^{\tau_f} d\tau \int_{\sigma_0}^{\sigma_1} d\sigma \delta X^\mu (\partial_\tau P^\sigma_\mu + \partial_\sigma P^\sigma_\mu) 
\]  

(3.10)

Since the second term on the right hand side of equation (3.10) must vanish for all variations, we get the following constraint:

\[
\frac{\partial P^\sigma_\mu}{\partial \tau} + \frac{\partial P^\sigma_\mu}{\partial \sigma} = 0 
\]  

(3.11)

More about the second term in this expression in the following sections. The first term deals with the string end points. There are two ways in which this term can be made to vanish. Either \(\delta X^\mu(\tau, \sigma^*) = 0\) at the string end points where \(\sigma^*\) is \(\sigma_1\) or 0. This is what is called the Dirichlet boundary condition. Otherwise, we can put \(P^\sigma_\mu(\tau, \sigma^*) = 0\). This is the free endpoint or the Neumann boundary condition. In string theory, we generically have open strings as well as closed strings. Obviously only open strings require boundary conditions. Also, as is clear from (3.10), we must fix a boundary condition for terms corresponding to each coordinate. In general, a mixture of Neumann and Dirichlet boundary conditions is chosen. For Dirichlet boundary conditions, the string end points are fixed in time and can therefore be imagined to be attached to some sort of a surface. This surface is what is called a D-brane. A Dp-brane is a brane with p spatial dimensions. If a Dirichlet boundary condition is imposed corresponding to a particular coordinate, the string is free to move in the directions tranverse to it and so we have a situation wherewith a string endpoints are free to move in the directions tangent to the D brane.

### 3.2 World volume gauge fields

As we showed earlier, D-branes provide surfaces to which open strings attach to. Consider a p+1 dimensional D brane (denoted as a Dp) brane in a d dimensional space. To specify these 'hyperplanes' we need (d-p) linear conditions. To see why this is the case, let us look at some results we already know: In three spatial dimensions where \(d = 3\), a 2 brane \((p = 2)\) is a plane. To specify it, we need \((d - p = 3 - 2 = 1)\)
one linear condition. For instance, $x = 0$ specifies the $y - z$ plane. Similarly, a piece of wire along the $z$ axis ($p = 1$) is specified by $(d - p = 3 - 1 = 2)$ two conditions: $x = 0$ and $y = 0$. To sum it up, we need as many conditions as there are spatial coordinates normal to the brane. Return now to the D$p$-brane.

Let us use $x^\mu$ to denote spacetime coordinates and let us split these coordinates into two groups. In the first group are the coordinates tangential to the brane which clearly are time and $p$ spatial coordinates since the brane is $p$-dimensional. The other $(d - p)$ coordinates are normal to the brane. The location of the brane, we know, is specified by the values of the coordinates normal to the brane. Let us denote these in the following manner:

$$x^a = \bar{x}^a, a = p + 1, ... d$$

Since we are looking at open strings that end on the $Dp$-brane, the coordinates normal to the brane must satisfy the Dirichlet boundary conditions, that is:

$$X^a(\sigma, \tau)|_{\sigma=0} = X^a(\sigma, \tau)|_{\sigma=\pi} = \bar{x}^a, a = p + 1, ..., d$$

where $X^a$ denote the string end-point coordinates. The string coordinates $X^a$ are referred to as the DD coordinates with the obvious reference to Dirichlet boundary conditions. Since the string is free to move along directions tangential to the brane, the coordinates of the endpoints satisfy the following equation:

$$X'^m(\sigma, \tau)|_{\sigma=0} = X'^m(\sigma, \tau)|_{\sigma=\pi} = 0, m = 0, 1, ..., p$$

Since these satisfy Neumann boundary conditions, these are referred to as the NN coordinates. To summarize, we can now split the string coordinates as

$$X^0, X^1, ..., X^p, X^{p+1}, X^{p+2}, ..., X^d$$

In terms of light-cone coordinates these coordinates are written as:

$$X^+, X^-, X^i, X^a$$

where $i = 2, ..., p$ and $a = p + 1, ..., d$

Having dealt with the preliminaries, let us now move to the problem of quantizing open strings on D branes. Since string theory is supposed to be a quantum theory, quantization is a requisite.
3.2. WORLD VOLUME GAUGE FIELDS

3.2.1 Quantizing open strings on Dp-branes

The Nambu-Goto action is re-parametrization invariant. Here we realize that we need gauge conditions to fix the parametrization of the world-sheets under consideration.

Fixing the parametrization

The gauge we will work with here is the following:

\[ n_\mu X^\mu(\tau, \sigma) = \lambda \tau \]  \hspace{1cm} (3.16)

One can show that this gauge fixes the strings to be curves located at the intersection of the world sheet with the hyperplanes orthogonal to \( n^\mu \). To see why this is, notice that for two points with the same value of \( \tau \), \( n_\mu (x_1^\mu - x_2^\mu) = 0 \). Therefore, any two points in the space defined by (3.4) lie in a place orthogonal to \( n^\mu \). It can be shown from the lagrangian of open strings with free endpoints that momentum is a conserved quantity. We can use this to restate our gauge condition as follows:

\[ n_\mu X^\mu(\tau, \sigma) = \tilde{\lambda}(n.p)\tau \]  \hspace{1cm} (3.17)

where we have replaced \( \lambda \) for another constant \( \tilde{\lambda}(n.p) \). We realize that \( n.X \) has units of length, \( n.p \) has units of momentum, \( \tau \) is a dimensionless number and therefore \( \tilde{\lambda} \) has dimensions of velocity. Therefore, in natural units, we set \( \tilde{\lambda} = 2\alpha' \) where \( \alpha' = l_s^2 \), the length of the string squared.

Having fixed the \( \tau \) parametrization, we now fix the \( \sigma \) parametrization. To do this, we require that the \( \sigma \) (Q) of the string be proportional to the integral of \( n.P^\tau \) over the portion of the string extending from \( \sigma = 0 \) to Q where \( P \) represents the momentum density of the string. Stated explicitly, we require that:

\[ (n.p)\sigma = \pi \int_{0}^{\sigma} d\tilde{\sigma} n.P^\tau(\tau, \tilde{\sigma}) \]  \hspace{1cm} (3.18)

Before moving on, we state that even though the above analysis holds only for open strings, it is easy to prove that these gauges also hold for closed strings up to some constant factors. In fact a detailed analysis leads to the following general result:

\[ n.X(\tau, \sigma) = \beta \alpha'(n.p)\tau; \]  \hspace{1cm} (3.19)

\[ (n.p)\sigma = \frac{2\pi}{\beta} \int_{0}^{\sigma} d\tilde{\sigma}(n.P^\tau(\tau, \tilde{\sigma})) \]  \hspace{1cm} (3.20)
where in the above $\beta = 2$ for open strings and $\beta = 1$ for closed strings. It can also be shown that the conditions given above lead to the following constraint on the coordinates:

$$ (\dot{X} \pm X')^2 = 0 \quad (3.21) $$

We now return to the equation of motion (3.11):

$$ \frac{\partial}{\partial \tau} (n.P^\tau) + \frac{\partial}{\partial \sigma} (n.P^\sigma) = 0 \quad (3.22) $$

where we have dotted both sides of the equation with $n$. Writing the infinitesimal form of the $\sigma$ gauge condition, we see that $n.P^\tau$ is a constant in time. Therefore, it follows that:

$$ \frac{\partial}{\partial \sigma} (n.P^\sigma) = 0 \quad (3.23) $$

It can be shown [2] that using the above constraints in parametrization the equation: $\frac{\partial}{\partial \tau} (P^{\tau\mu}) + \frac{\partial}{\partial \sigma} (P^{\sigma\mu}) = 0$ takes the form

$$ \ddot{X}^\mu - X'^{''\mu} = 0 \quad (3.24) $$

which is the usual wave equation. As a result, we conclude that our string endpoints satisfy the wave equation.

**Solutions of the wave equation**

We recall here that the free endpoint boundary conditions imply that $\frac{\partial X^\mu}{\partial \sigma} = 0$ at $\sigma = 0, \pi$. The first condition means that the solution to our wave equation must acquire the form:

$$ X^\mu(\tau, \sigma) = \frac{1}{2} (f^\mu(\tau + \sigma) + f^\mu(\tau - \sigma)) \quad (3.25) $$

The second condition implies the following:

$$ \frac{\partial X^\mu}{\partial \sigma}(\tau, \sigma) = \frac{1}{2} (f'^\mu(\tau + \pi) - f'^\mu(\tau - \sigma)) = 0 \quad (3.26) $$

This equation tells us that $f'^\mu$ is a periodic function with period $2\pi$. We can therefore write a general fourier series solution to the equation above in the following manner:

$$ f'^\mu(u) = f'^\mu_1 + \sum_{n=1}^{\infty} (a_n^\mu \cos(nu) + b_n^\mu \sin(nu)) \quad (3.27) $$
3.2. WORLD VOLUME GAUGE FIELDS

We integrate this equation to get

\[ f^\mu(u) = f^\mu_0 + f^\mu_1 + \sum_{n=1}^{\infty} (A^\mu_n \cos(nu) + B^\mu_n \sin(nu)) \]

where the constants of integration have been absorbed the constants that came about as a result of the integration into our new definition of the constants. We now substitute this expression into equation (3.10) to get:

\[ X^\mu(\tau, \sigma) = f^\mu_0 + f^\mu_1 \tau + \sum_{n=1}^{\infty} (A^\mu_n \cos(n\tau) + B^\mu_n \sin(n\tau)) \cos(n\sigma) \]

We use the Euler identity to write

\[ A^\mu_n \cos(n\tau) + B^\mu_n \sin(n\tau) = -\frac{l}{2}((B^\mu_n + iA^\mu_n)e^{in\tau} - (B^\mu_n - iA^\mu_n)e^{-in\tau}) = -\frac{\sqrt{2\alpha'}}{\sqrt{n}}(a^\mu_n e^{in\tau} - a^\mu_n e^{-in\tau}) \]

(3.30)

Where in the last step, we have defined new, dimensionless constants \( a^\mu_n \). We then use the definitions of momentum density for the string \( P^\tau \mu = \frac{1}{2\pi\alpha'} \hat{X}^\mu \) to evaluate it in terms of \( f^\mu_n \) from the expression (3.10). Also, since \( p^\mu = \int_0^\pi P^\tau \mu d\sigma \) it turns out that \( f^\mu_1 = 2\alpha' p^\mu \). Fixing \( f^\mu_0 = x^\mu_0 \) we get:

\[ X^\mu(\tau, \sigma) = x^\mu_0 + 2\alpha' p^\mu \tau - \frac{\sqrt{2\alpha'}}{\sqrt{n}} \sum_{n=1}^{\infty} (a^\mu_n e^{in\tau} - a^\mu_n e^{-in\tau}) \cos(n\sigma) \]

(3.31)

One can see that the write hand side has a term each corresponding to the zero mode, to the linear momentum and to oscillations of the string. Just for the sake of convenience later on, we make the following definitions:

\[ \alpha^\mu_0 \sqrt{2\alpha'} p^\mu \alpha^\mu_n = a^\mu_n \sqrt{n}\alpha^\mu_{-n} = a^{*\mu}_n \sqrt{n}, n \geq 1 \]

(3.32)

Notice that \( \alpha^\mu_n = (\alpha^\mu_{-n})^* \)

We use these to now write a compact version of (3.16)

\[ X^\mu(\tau, \sigma) = x^\mu_0 + \sqrt{2\alpha'} \alpha^\mu_0 \tau + \sqrt{2\alpha'} \sum_{n \neq 0}^{\infty} \frac{1}{n} \alpha^\mu_n e^{-in\tau} \cos(n\sigma) \]

(3.33)

Again for later convenience, we calculate and write the following expression:

\[ \dot{X}^\mu \pm X^\mu = \sqrt{2\alpha'} \sum_{n \in \mathbb{Z}} \alpha^\mu_n e^{-in(\tau \pm \sigma)} \]

(3.34)
The equations of motion and indeed the conditions (3.7) and (3.8) can be cast into a very convenient form using light cone coordinates. For the sake of avoiding further digression we will skip the detailed analysis and just state the important results. The interested reader is referred to [2] to fill them in.

One very important result here is the expression for the mass of the string. Using the relativistic equations of mass, we know that in natural units $p.p = -m^2$. One can show that in light cone coordinates $2p^+p^- - p^i p^i = -p^2 = M^2$ which can be written in the following way:

$$M^2 = \frac{1}{\alpha'} \sum_{n=1}^{\infty} na^*_n a_n$$ (3.35)

Towards quantization

In light-cone coordinates, we have the following equation [2] for string coordinates:

$$\dot{X}^± ± X^±' = \frac{1}{2\alpha'} \frac{1}{2p^+}(\dot{X}^I ± X^I')^2$$ (3.36)

The mode expansion of the transverse coordinates is given by:

$$\dot{X}^I ± X^I' = \sqrt{2\alpha'} \sum_{n \in \mathbb{Z}} \alpha^I_n e^{-\imath n(r±\sigma)}$$ (3.37)

This leads to the following expansion:

$$2p^+p^- = \frac{1}{\alpha'} (\frac{1}{2} \alpha_0^I \alpha_0^I + \sum_{n=1}^{\infty} \alpha^I_{-n} \alpha^I_n + a)$$ (3.38)

where 'a' is a constant that comes about as a result of normal ordering. Splitting up the relation (3.25) in terms of DD coordinates labeled by a and NN coordinates labeled by i, we get:

$$\dot{X}^± ± X^±' = \frac{1}{2\alpha'} \frac{1}{2p^+}((\dot{X}^i ± X^i')^2 + (\dot{X}^a ± X^a')^2)$$ (3.39)

Having already showed that the coordinates $X^a$ normal to the brane satisfy the wave equation, we now proceed to right the expansion of $X^a$ in terms of a fourier series. Using the boundary conditions (3.1) and (3.2) we write the following expansion for $X^a$:
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\[ X^a(\tau, \sigma) = \bar{x}^a + \sum_{n=1}^{\infty} (f_n^a \cos(n\tau) + \tilde{f}_n^a \sin(n\tau)) \sin(n\sigma) \]  

(3.40)

To quantize this, we must turn the expansion coefficients, \((f^a, \tilde{f}^a)\) into operators. Repeating the analysis of equations (3.19), (3.20) and (3.21) we get:

\[ X^a(\tau, \sigma) = \bar{x}^a + \sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha_n^a e^{-in\tau} \sin(n\sigma) \]  

(3.41)

Using the quantization prescription:

\[ [X^a(\tau, \sigma), \dot{X}^b(\tau, \sigma')] = 2\pi\alpha' \iota \delta^{ab} \delta(\sigma - \sigma') \]  

(3.42)

we get:

\[ [\alpha_m^a, \alpha_n^b] = m\delta^{ab} \delta_{m+n,0}, m, n \neq 0 \]  

(3.43)

Using this and the fact the normal ordering constant ”\(a\)” can be shown to be -1 for a D-25 brane (which is what we consider for now), we write:

\[ 2p^+ p^- = \frac{1}{\alpha'} (\alpha' p^i p^i + \sum_{n=1}^{\infty} \left[ \alpha_n^{-1} \alpha_n^i + \alpha_n^a \alpha_n^a \right] - 1) \]  

(3.44)

Using \(M^2 = -p^2 = 2p^+ p^- - p^i p^i\) and re-introducing the creation and annihilation operators in place of \(\alpha's\) we get the following important relation:

\[ M^2 = \frac{1}{\alpha'} (-1 + \sum_{n=1}^{\infty} \sum_{i=2}^{p} n a_n^{ai} a_n^{ai} + \sum_{m=1}^{d} \sum_{a=p+1}^{d} m a_m^{ai} a_m^{ai}) \]  

(3.45)

This operator gives rise to particle states. Consider the state which is annihilated by the creation operators, the ground state. It has \(M^2 = \frac{1}{\alpha'}\) Now this implies an imaginary mass. Such states are called tachyonic states and are a regular feature of bosonic string theories. In fact one of the main cases for supersymmetric string theories is the fact that they get rid of these unphysical tachyonic states. We will show in a moment, however, how we can get rid of them within this framework as well. The next states have \(a^i\) acting on them once from the second term of the expression. These terms have \(M^2 = 0\). From the indices it is clear that there are \(p - 1\) of these states. Since the index they carry lives on the brane they transform as a lorentz vector on the brane. Since the number of states is equal to the space time dimensions of the brane.
minus 2, we conclude that these are photon states. The associated field is therefore a Maxwell gauge field which lives on the brane. Consider now the action of creation operators in the third term of the expression. Acting once, the consequent states have $M^2 = 0$. However, these are not labelled by indices that are lorentz indices on the brane. Instead these are just counting labels. Hence these transform as scalars. As a result, we have two very important results: 1) A Dp-brane has a Maxwell field on its world volume and 2) A Dp-brane has a massless scalar for each direction normal to it.

We can now repeat the above analysis for the case where a string is stretched between two different branes, that is, with one end point on each. Based on that, the expression for $M^2$ can be obtained similarly. It turns out that now:

$$M^2 = \left(\frac{x_2^a - x_1^a}{2\pi\alpha'}\right)^2 + \frac{1}{\alpha'}(-1 + \sum_{n=1}^{\infty} \sum_{i=2}^{p} na_n a_i^i + \sum_{m=1}^{\infty} \sum_{a=p+1}^{d} ma_m a_m)$$ (3.46)

Here the branes under consideration are assumed to be parallel and to have the same dimensions. Moreover, $x^a = \bar{x}_1^a$ specifies the location of the first Dp-brane and $x^a = \bar{x}_2^a$ of the second one. The ground state now has $M^2 = \frac{-1}{\alpha'} + \left(\frac{x_2^a - x_1^a}{2\pi\alpha'}\right)^2$ Hence, if the two branes are separated by $|\bar{x}_2^a - \bar{x}_1^a| = 2\pi\sqrt{\alpha'}$ or more, the ground state is no more tachyonic. A tachyonic ground state corresponds to an instability of the D-brane. The negative mass squared implies a downward opening parabolic-potential. It can be shown that it leads to a decay of the D-brane which is why tachyons are a source of instability which is why we want to avoid them.

The first excited states arise in two ways as before. From the third part of the second term we get $d - p$ states with $M^2 = \left(\frac{x_2^a - x_1^a}{2\pi\alpha'}\right)^2$. Again, since $a$ is not a lorentz index on the brane, these are $d - p$ massive scalar fields. States with the same mass also arise from the second part of the second term. This time, these indices are lorentz indices on the brane are there are $p - 1$ of these states. However, unlike the previous case, these are not massless. It can be shown that in a D-dimensional spacetime, massive gauge fields have $D - 1$ states and therefore, one of the scalar states must join these $p - 1$ states to a form a massive vector. All in all, there is one massive vector field and $d - p - 1$ scalars.

In the limit that the separation of the two branes goes to zero, there are 4 open string sectors: strings can either have one end point on each of the branes or both end
3.3 String and brane charges

A charged point particle couples to a Maxwell field. It has a one-dimensional world volume and hence just one tangent vector. The maxwell field $A_\mu$ also carried one index and so we can get a Lorentz scalar from the terms $A_\mu \frac{dx^\mu}{d\tau}$ where $\tau$ parametrizes the world volume of the particle. Indeed the interaction of a charged point particle with a maxwell field is given by the following lagrangian:

$$q \int A_\mu \frac{dx^\mu}{d\tau} d\tau \quad (3.47)$$

This index matching approach to predicting candidate lagrangians in extremely important when writing down actions for multi-dimensional things such as branes.

---

1 The U(1) mode represents the free motion of the brane. We will only be concerned with the modes that couple to gravity and since the rest will be discarded so we can safely start ignoring U(1).
and strings. This is exactly what we will employ here. A string traces out a two dimensional world volume or a world sheet. Naturally, since it has two vectors that span the sheet, we would expect there to be a two form field to multiply the tangent vectors to give a lorentz scalar, or in more relevant terms, the lagrangian. Hence, we should have something of the form:

$$-\int d\tau d\sigma \frac{\partial X^\mu}{\partial \tau} \frac{\partial X^\nu}{\partial \sigma} B_{\mu\nu}(X(\tau, \sigma))$$ (3.48)

where $B_{\mu\nu}$ is a two form called the Kalb-Ramond field. It can be shown that to ensure re-parametrization invariance of the string action, $B_{\mu\nu}$ must be antisymmetric. For the sake of completeness we will mention here the corresponding field strength $H_{\mu\nu\rho}$ which is a generalization of the maxwell field strength $F_{\mu\nu}$. Since $B_{\mu\nu}$ is the only thing that will be used in this thesis, the details of some of the other accompanying concepts including $H_{\mu\nu\rho}$ are skipped and the interested reader is referred to [2].

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$$ (3.49)

Having obtained the required generalization for the case of strings, let us now move on to branes. A Dp-brane has a world volume which is $p+1$ dimensional. One would then expect branes to couple to to $p+1$ forms. These are called Ramond-Ramond (RR) forms. Indeed the coupling term in the action is given by the following:

$$S = -\int d\tau d\sigma_1...d\sigma_p \frac{\partial X^\mu}{\partial \tau} \frac{\partial X^{\mu_1}}{\partial \sigma_1}...\frac{\partial X^{\mu_p}}{\partial \sigma_p} A_{\mu_1...\mu_p}(X(\tau, \sigma_1,...\sigma_p))$$ (3.50)

As a result we now see that like point particles, strings and branes also carry charges. It is as a result of charge and energy conservation that charged branes, for example, cannot decay if there are no lighter decay products that carry charge. Hence, charged branes are more stable than uncharged branes. Since we will always be interested in studying only the stable structures, we will only be concerned with strings and branes that carry charges and couple to the corresponding fields.

### 3.3.1 Strings ending on D-branes

The notion of charge conservation is related to gauge invariance. To understand this consider the following: In electromagnetism, the lagrangian has a term: $q \vec{v}.\vec{A}$ which shows that the the vector potential couples to the current. The coupling term in the action hence has the form:
3.3. STRING AND BRANE CHARGES

\[ S = \int d^D x A_\mu(x) j^\mu(x) \]  

(3.51)

The gauge transformation, or the transformation that leaves the field strength invariant is:

\[ \delta A_\mu = \partial_\mu \epsilon \]  

(3.52)

Under this transformation, the whole maxwell lagrangian in invariant, not just the coupling term. Now under this transformation:

\[ \delta S_{\text{coup}} = \int d^D x (\partial_\mu \epsilon) j^\mu(x) = -\int d^D x \epsilon \partial_\mu j^\mu(x) \]  

(3.53)

where the last equality is obtained after integrating by parts and assuming that at the boundary the variation vanishes. Hence, gauge invariance: \( \delta S_{\text{coup}} = 0 \) follows from current conservation \( \partial_\mu j^\mu = 0 \). This can now be generalized to the Kalb-Ramond field where:

\[ \delta B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \]  

(3.54)

and the coupling term in the action is given by:

\[ -\int d^D x B_{\mu\nu}(x) j^{\mu\nu}(x) \]  

(3.55)

Now, it turns out that in order to make the string action gauge invariant for the case when the string endpoints lie on branes, we must add terms to it in the following manner:

\[ S = S_{\text{original}} + \int d\tau A_m(X) \frac{dX^m}{d\tau} |_{\sigma=\pi} - \int d\tau A_m(X) \frac{dX^m}{d\tau} |_{\sigma=0} \]  

(3.56)

The additional two terms correspond to the string endpoints and since these couple to the maxwell field with opposite signs, we conclude that string endpoints are oppositely, electrically charged. The relevant gauge transformation in this case is given by:

\[ \delta B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \]  

(3.57)

\[ \delta A_m = -\Lambda_m \]  

(3.58)
Under this transformation, mind you,

\[ \delta F_{mn} = \partial_m \delta A_n - \partial_n \delta A_m = -\delta B_{mn} \]  

(3.59)

Hence now \( F_{mn} \) is no longer gauge invariant. The gauge invariant term now, in fact, is:

\[ F_{mn} = F_{mn} + B_{mn} \]

This is the term that will figure in the Dirac Born Infeld action which is what we study next.

### 3.4 Born-Infeld electrodynamics and DBI action

In this section, we will digress a little to discuss non linear electrodynamics, in particular, Born-Infeld electrodynamics in an attempt to motivate how the DBI action for branes comes about. That will conclude our discussion of D-branes.

#### 3.4.1 The framework

We are most used to studying the maxwellian theory of electrodynamics which is a linear theory, that is, in free space or in homogeneous and isotropic materials, we can define quantities \( D \) and \( H \) which are linearly related to \( E \) and \( B \):

\[ B = \mu HE = \epsilon E \]  

(3.60)

where \( \mu \) and \( \epsilon \) are scalars. These lead us to a set of linear, partial differential equations called the maxwell’s equations. However we know that this theory has some problems as well, one of which is the infinite self energy of a point charge. In addition, in a relativistic theory of electrodynamics, there is an upper bound on the allowed value of electric field. This upper bound comes about as a result of T-duality. We will skip the details of T-duality for now and will restrict myself to just saying that according to T-duality, a D-brane with an electric field is equivalent to a D-brane that is moving. Since there is an upper bound on the speed at which anything can move, that is, the speed of light, an upper bound on the value of the electric field comes about naturally. Therefore, we need a new theory that takes care of that. Of
course, since maxwell’s theory is so well tested, our theory must also reduce to it in
a certain limit.

Let us go about constructing the lagrangian for this theory step by step. Our
first requirement is the existence of a maximum value for the electric field. Let’s say
$E \leq b$. Consider:

$$
\mathcal{L} = -b^2 \sqrt{1 - \frac{(E^2 - B^2)}{b^2}} + b^2
$$

(3.61)

At $B = 0$ $E \leq b$ otherwise the argument of the radical is negative. Also, if
$s = E^2 - B^2$ is small:

$$
\mathcal{L} \approx -b^2 (1 - \frac{s}{b^2}) + b^2 \approx s = E^2 - B^2
$$

(3.62)

This is just the maxwell lagrangian. Let us use this as a base to construct a more
complicated lagrangian. Consider:

$$
\mathcal{L} = -b^2 \sqrt{1 - \frac{E^2 - B^2}{b^2} - \frac{(\vec{E} \cdot \vec{B})^2}{b^4}} + b^2
$$

(3.63)

Since it is constructed out of scalars, the lagrangian is manifestly lorentz invariant.
Moreover, when $E^2 - B^2$ and $(\vec{E} \cdot \vec{B})^2$ are much smaller than $b^2$, we recover the maxwell
lagrangian. The special thing about this lagrangian is that it can easily be rewritten
in a form where a generalization to any number of dimensions becomes possible. This
is:

$$
\mathcal{L} = -b^2 \sqrt{-\text{det}(\eta_{\mu\nu} + \frac{1}{b} F_{\mu\nu})} + b^2
$$

(3.64)

It can be shown that this theory leads a finite value of self energy for a point
charge, the reader can look up the derivation in chapter 20 of [2]. Since the ends of
our strings on D-branes are point charges, much to our relief, there are no infinite
energies over there, as per this theory.

### 3.4.2 Compatibility with T-duality

String theory offers a lot of extra dimensions. We must find a way to deal with these
extra dimensions in our theories. One of these ways is to compactify these dimen-
sions. This compactification, amazingly, leads to a symmetry: If a spatial dimension
is curled up into a circle of radius $R$ then the scenario is physically equivalent to a scenario where the the radius is $\propto \frac{1}{R}$. This is what is called T-duality.

By imposing the conditions for T-duality, we get a constraint on the lagrangian proposed in the previous subsection. It turns out that the lagrangian has to be of the following form:

$$L \propto \sqrt{-\det(\eta_{\mu\nu} + 2\pi \alpha' F_{\mu\nu})}$$ \hspace{1cm} (3.65)

where $F_{\mu\nu}$ has been replaced with $F_{\mu\nu}$ preempting the fact that we will be concerned with a background field in this thesis and as already mentioned earlier, the gauge invariant term in that case is not $F_{\mu\nu}$ but $F_{\mu\nu}$. T-duality also provides confirmation that this lagrangian does really describe D-branes.

This is what is called the Dirac-Born-Infeld action for D-branes. This is primarily what we will be making use of from here on.
Chapter 4

The AdS/CFT correspondence

Having discussed the basics, we now move to the main topic of this thesis: The AdS/CFT correspondence. The discussion in this chapter borrows from [3]. We will attempt to motivate how the correspondence comes about in order to provide a background for the things in the following chapters. Details of the AdS/CFT dictionary will be skipped and the interested reader is referred to [3] to study them.

4.1 The motivation

In its most general form, the correspondence says that a theory of gravity in the bulk of a space is equivalent to a gauge theory on the boundary of the space. By equivalent we mean that they describe the same physics and hence there is a map between operators in the conformal field theory theory and fields in supergravity.

Consider a stack of $N$ parallel, coincident $D3$–branes in 10 dimensional space-time. They lie along a $(3+1)$ dimensional hyperplane and are all located at the same point in the transverse 6 dimensional space. In this background, there are two types of string excitations: the open string excitations, as we studied in the previous chapter, correspond to the $D3$–branes while the closed string excitations are of the 10 dimensional ambient space. The open string excitations are described by a $U(N) \rightarrow SU(N)$ theory as we saw earlier. It can be shown that at low energies, the complete description of open strings is given by an $\mathcal{N} = 4, SU(N)$ supersymmetric theory. One can additionally also show that the closed string excitations are described by type IIB
supergravity in 10 dimensional flat space. At low energies, the interaction term between the closed strings and open strings can be ignored. Hence, the low energy limit of a stack of D3-branes in 10 dimensions is described by two independent sectors: \( N = 4, SU(N) \), a 4 dimensional conformal field theory and by type IIB supergravity in 10 dimensional flat space.

Let us now look at the same configuration by considering how D3-branes effect the 10 dimensional ambient spacetime at low energies by looking at the corresponding solutions to the equations of supergravity. The metric is given by:

\[
\begin{align*}
\text{ds}^2 & = H^{-rac{1}{2}} (-dt^2 + d\vec{x}^2) + H^{rac{1}{2}} (dr^2 + r^2 d\Omega_5^2) \\
\end{align*}
\]

where

\[
H = 1 + \frac{4\pi g_N \alpha'}{r^4}
\]

Now consider an observer sitting at \( r = \infty \). Corresponding to the energy \( E_r \) at any radial distance \( r \), the energy that he measures is given in \([3]\) :

\[
E_{\infty} = H^{-rac{1}{2}} E_r
\]

Consequently, from the perspective of the observer at infinity, any object moving towards \( r = 0 \) (the near horizon region) regardless of its 'actual' energy, would appear to have lower and lower energy because of red shift. However, that observer also sees low energy excitations arising from the standard large wavelength excitations, that is, energies that are already low even before the red shift. At low energies, these two excitations decouple (it is possible to show that the coupling goes like \( \approx L^8 \omega^3 \)). The overall description therefore, at low energies, is given by two sets of non interacting excitations: ones in the near horizon region and ones far away from the branes where the space is essentially flat and is described by free, 10 dimensional type IIB supergravity. In the limit that \( r \to 0 \) \(^1\) (4.1) becomes \([3]\) :

\[
\begin{align*}
\text{ds}^2 & = \frac{L^2}{r^2} dr^2 + \frac{r^2}{L^2} (-dt^2 + d\vec{x}^2) + L^2 d\Omega_5^2 \\
\end{align*}
\]

where \( L = (4\pi g N \alpha'')^{rac{1}{4}} \)

Defining, a new radial coordinate \( u = \frac{L^2}{r} \) this becomes:

\[
\begin{align*}
\text{ds}^2 & = \frac{L^2}{u^2} [du^2 - dt^2 + d\vec{x}^2] + L^2 d\Omega_5^2 \\
\end{align*}
\]

\(^1\) The near horizon limit is alternatively also taken by taking \( \alpha' \to 0 \) while keeping \( \frac{r}{\alpha'} \) fixed
4.2. THE 'T'HOOF LIMIT

This metric describes the space $AdS_5 \times S_5$

In both of the descriptions of the above system, we found that the system was described by two decoupled sub systems and in both cases, one of these was free 10 dimensional type IIB supergravity. One is therefore led to conclude that the description carried by the other sub system in each of the above cases is equivalent, that is in the low energy limit:

$$\mathcal{N} = 4, U(N) \text{ super Yang-Mills theory in 4 dimensional minkowski space is equivalent to type IIB supergravity in } AdS_5 \times S_5$$

This is the AdS/CFT conjecture.

4.1.1 Evidence from symmetry matching

A conformal field theory in 4 dimensions has conformal symmetries that are generated by a set of 15 operators. These operators define the conformal lie algebra which contains the lorentz algebra as a sub-algebra. The generators of these transformations, as shown in chapter 1, satisfy the SO(4,2) commutation relations. It follows from our discussion of AdS spaces in chapter 2 that the isometric group of $AdS_5$. Moreover, $\mathcal{N} = 4, SU(N)$ has an R-symmetry group SO(6) which matches the isometry group of $S^5$. As a result, the symmetry groups of the two theories match providing further evidence of a correspondence between the two.

The AdS/CFT correspondence provides us with a powerful tool to study gauge theories which are difficult to study on their own using theories of supergravity. In fact, in any given situation, we can make use of the duality to study a particular problem using either one of the two theories, whichever is easier in the particular context.

4.2 The t’hooft limit

One useful limit in which the AdS/CFT conjecture becomes particularly handy is called the t’hooft limit, wherein the N in $SU(N)$ or the number of colors tends to infinity while $\lambda = g_{ym}^2 N$ or the effective gauge coupling parameter is kept constant.

$$2N = \frac{(d+2)(d+1)}{2}$$ as shown in chapter 1
The coupling in the dual gravity picture is $g \propto \frac{1}{N^2}$ and hence large $N$ implies that the gauge theory is strongly coupled but the dual gravity theory is weakly coupled and therefore easy to study. It has been a long standing problem to study strongly coupled gauge theories. If understood, newer and better insights into phenomena such as quark confinement would become possible. The usual approach through perturbation theory fails here precisely because the coupling constants are large and higher order Feynman diagrams contribute more and more. Using the AdS/CFT correspondence, however, the analysis can be done through the dual gravity theory which is weakly coupled and hence more tractable. This is exactly what we proceed to do next.
Chapter 5

Large N flavored gauge theories on a compact manifold: The Background

The problem we are mainly interested in addressing is of quark confinement. We discussed at the end of the last chapter how the large N limit potentially allows us to study it using the AdS/CFT correspondence. We will make full use of the correspondence to study everything in the dual gravity framework and use the AdS/CFT dictionary mentioned in [10] to allow us to do that.

The world volume of a stack of D3-branes, we saw in the last chapter, naturally defines an $\mathcal{N} = 4$SYM. We also discussed how in the near horizon limit, the D3-brane solution to ten dimensional supergravity becomes equivalent to $AdS_5 \times S^5$. Hence, in this limit, a stack of D3-branes can be replaced by $AdS_5 \times S^5$ and vice versa.

The AdS/CFT dictionary relates the embedding of the brane to the vacuum expectation value of the corresponding gauge theory. More on this a little later. In addition, the fluctuations of the embedding correspond to the meson spectrum of the gauge theory [10]. In our discussion of D-branes, we talked about how open strings ending on D-branes lead to particle states upon quantization which define a gauge theory on the world volume. It can be shown that the endpoints of strings which lie on the same stack of D-branes transform in the adjoint representation of the corresponding gauge group. As a result, to introduce fundamental matter we need to introduce another stack of D-branes on which one of the endpoints lies. Placing it in the transverse di-
Reections to the original stack adds a global symmetry to our theory which is referred to as flavor. The fundamental matter in our case is obviously quarks. As discussed before, extending the stings between two stacks of branes gives them mass. In fact, the mass of the quarks is given in [10]:

\[ m_q = \frac{L}{2\pi \alpha'} \]  

(5.1)

where \( L \) is the asymptotic separation between the two stacks and \( \alpha' \) is related to the string tension. This is how massive quarks come about in the dual gravity picture. Furthermore, adding a stack of Dq-branes where \( q = p + 4 \) (\( p \) denotes the number of spatial dimensions of our original stack which in this case is 3), it can be shown, breaks half the supersymmetry and gives us an \( \mathcal{N} = 2 \) theory [7]. The addition of flavor here allows us to mimic realistic QCD which has flavors in it.

The addition of flavor branes, D7-branes in our case, has a back reaction on the geometry. It is equivalent to \( AdS_5 \times S_5 \) in the near horizon limit when we only have a stack of D3-branes. In order to be able to use the correspondence, we obviously need our space to remain \( AdS_5 \times S_5 \) and hence we impose that the number of D7-branes, \( N_f \) is much much less than the number of D3-branes, \( N_c \). This corresponds to the quenched approximation in lattice QCD.

The fact that our fermionic fields exist in a compact space \( (AdS_5 \times S_5) \) leads to a repulsive casimir force. For the case of a sphere, it is \( \propto \frac{1}{R} \) [6]. Applying an external magnetic field counters the dissociating effect of this and the competition of these two causes a confinement/deconfinement phase transition which is primarily what we will be interested in studying in the next chapter.

Before moving on, we will digress a little to discuss chiral symmetry breaking. We will see in the next chapter that an external magnetic field leads to chiral symmetry breaking which is when the vacuum expectation value becomes non zero even at zero quark mass.
5.1 Spontaneous breaking of chiral symmetry

The term in the $SU(N_c)$ lagrangian coupled to $N_f$ flavors describing fundamental fields is given by [10]:

$$L_f = \sum_{i=1}^{N_f} \left( \Psi_L^i \gamma^\mu D_\mu \Psi_L^i + \Psi_R^i \gamma^\mu D_\mu \Psi_R^i - m_i (\Psi_L^i \Psi_R^i + \bar{\Psi}_L^i \Psi_L^i) \right)$$

(5.2)

where $\Psi_{L/R} = \frac{1}{2} (1 \pm \gamma^5) \Psi$

At zero mass, left and right handed spinors transform independently since the coupling term only comes about if mass is non-zero— we say that a chiral symmetry exists. This global transformation is given by: $U(N_f)_R \times U(N_f)_L$.

Remember that $U(N) = SU(N) \times U(1)$. An infinitesimal $U(1)$ transformation is given by:

$$\delta \Psi_L = -i\alpha \Psi_L; \delta \Psi_R = i\alpha \Psi_R;$$

(5.3)

Now:

$$\Psi_L + \Psi_R = \Psi \implies \delta \Psi = -i\alpha \Psi_L + i\alpha \Psi_R \implies \delta \Psi = -i\alpha \gamma^5 \Psi$$

$$\delta \Psi = -i\alpha \gamma^5 \Psi \implies \delta \Psi = (\gamma^5 \Psi)^\dagger \gamma^0 \implies \delta \Psi = i\alpha (\gamma^5 \Psi)^\dagger \gamma^0 = -i\alpha \Psi^\dagger \gamma^0 \gamma^5 = -i\alpha \bar{\Psi} \gamma^5$$

The conserved current is given by:

$$j^\mu = \frac{\partial L}{\partial (\partial_\mu \Psi)} (-i\alpha \gamma^5 \Psi)$$

(5.4)

which is:

$$j^\mu = \bar{\Psi} \gamma^\mu \gamma^5 \Psi$$

(5.5)

Now consider the following: In order for the above transformation to remain a symmetry of the system,

$$\langle \bar{\Psi} \Psi \rangle$$

(5.6)

must remain invariant under

---

1The complete lagrangian is given in [10] but the other terms are irrelevant here.

2We could alternatively also have looked at the SU(N) part of the symmetry but the U(1) part is what we are interested in breaking in this thesis.
\[ \Psi \rightarrow \Psi - i\alpha \gamma^5 \Psi \]

and

\[ \bar{\Psi} \rightarrow -i\alpha \bar{\Psi} \gamma^5 + \bar{\Psi} \]

so that the expectation value of the operator \( \bar{\Psi} \Psi \) in the vacuum remains the same. This imposes the condition that:

\[ \langle \bar{\Psi} \Psi \rangle = \langle \bar{\Psi} \Psi - 2i\alpha \bar{\Psi} \gamma^5 \Psi - \alpha^2 \Psi \bar{\Psi} \rangle \quad (5.7) \]

Since the equation must hold for arbitrary \( \alpha \) we conclude that \( \langle \bar{\Psi} \Psi \rangle \) must vanish. Note that when the expectation value of \( \langle \bar{\Psi} \Psi \rangle \) in the ground state is non zero, the overall \( U(N_f)_R \times U(N_f)_L \) is still a symmetry of the lagrangian but no longer a symmetry of the ground state. This is what is called spontaneous symmetry breaking.

Hence, \( \langle \bar{\Psi} \Psi \rangle \) is an order parameter. When this vacuum expectation value (vev) is non zero, we conclude that chiral symmetry is spontaneously broken at zero mass. We will use this fact in the next chapter to study chiral symmetry breaking using the AdS/CFT dictionary which tells us that this order parameter is related to the brane embedding.
Chapter 6

Large N flavored gauge theory on a compact manifold with an external magnetic field: The application.

In this chapter we review the discussion and reproduce the computations done in [4] in order to demonstrate how the AdS/CFT correspondence can be used to study strongly coupled gauge theories.

6.1 The setting

In (2.14) we discussed the generic form of the metric for an AdS space. Defining \( \sinh(\mu) = \frac{r}{L} \) in that expression, we get the following metric for AdS\(_5\):

\[
ds^2 = -(1 + \frac{r^2}{R^2})d\tau^2 + r^2 d\Omega_3^2 + \frac{dr^2}{1 + \frac{r^2}{R^2}} \tag{6.1}
\]

where \( R^2 dt^2 = d\tau^2 \)

The metric, therefore, for AdS\(_5\) \( \times S^5 \) is given by:

\[
ds^2 = -(1 + \frac{r^2}{R^2})d\tau^2 + r^2 d\Omega_3^{(1)2} + \frac{dr^2}{1 + \frac{r^2}{R^2}} + R^2 d\Omega_5^2 \tag{6.2}
\]

where the superscript \( (1) \) on \( d\Omega_3^2 \) is put to differentiate it from the solid angle of the 5-sphere. Here:

\[
d\Omega_5^2 = d\theta_3^2 + \cos^2 \theta_3 d\Omega_3^{(2)2} + \sin^2 \theta_3 d\phi_3^2 \tag{6.3}
\]
\[ d\Omega_3^{(1)2} = e^{(1)2} + e^{(2)2} + e^{(3)2} \]  

(6.4)

where

\[ e^{(1)} = d\theta_1, e^{(2)} = \sin \theta_1 d\phi_1, e^{(3)} = \cos \theta_1 d\psi_1 \]

As a shorthand, these coordinates will from now on be referred to as 1, 2 and 3 instead of \( e^{(1)}, e^{(2)} \) and \( e^{(3)} \). Also:

\[ d\Omega_3^{(2)2} = d\psi^2 + \cos^2 \psi d\beta^2 + \sin^2 \psi d\gamma^2 \]  

(6.5)

The angular coordinates on the 5-sphere are labeled with subscript 3, while the coordinates on \( S^3 \subset AdS_5 \) are given the subscript 1. The remaining two coordinates, \( \tau \) and \( r \) specify the time like coordinate and the radial coordinate on \( AdS_5 \) respectively. These 10 coordinates in all make up our 10 dimensional space: \( AdS_5 \times S^5 \)

Let us now redefine the radial coordinate \( r \) as follows:

\[ u = \frac{1}{2} \left( r + \sqrt{R^2 + r^2} \right) \]  

(6.6)

The claim is that the metric (6.2) under this redefinition becomes:

\[ ds^2 = -\frac{u^2}{R^2} \left( 1 + \frac{R^2}{4u^2} \right)^2 d\tau^2 + u^2 \left( 1 - \frac{R^2}{4u^2} \right)^2 d\Omega_3^{(1)2} + \frac{R^2}{u^2} \left( du^2 + u^2 d\Omega_5^2 \right) \]  

(6.7)

Let us verify this claim. Consider the first term in (6.6).

\[ -\frac{u^2}{R^2} \left( 1 + \frac{R^2}{4u^2} \right)^2 = -\frac{1}{4R^2} \left( \frac{2\sqrt{R^2+r^2}(2r^2+2R^2+2r\sqrt{R^2+r^2})}{2r^2+2R^2+2r\sqrt{R^2+r^2}} \right)^2 \]

\[ \Rightarrow -\frac{1}{4R^2} \left( \frac{4(R^2+r^2)}{4u^2} \right)^2 \]

\[ \Rightarrow -(1 + \frac{r^2}{R^2}) \]

Now consider the second term in (6.6):

\[ u^2 \left( \frac{4u^2-R^2}{4u^2} \right)^2 = u^2 \left( \frac{2r^2+2r\sqrt{R^2+r^2}}{4u^2} \right)^2 = u^2 \left( \frac{4r}{4u^2} \right)^2 = r^2 \]

The third term in (6.6) has two parts. The angular part is trivially equal to \( R^2 \). The other part is given by:
6.1. THE SETTING

\[ \frac{R^2}{u^2} du^2 = \frac{R^2}{u^2} (\frac{dr}{2} + \frac{r dr}{2 \sqrt{R^2 + r^2}})^2 = \frac{R^2}{u^2} (\frac{2u}{2 \sqrt{R^2 + r^2}})^2 dr^2 = \frac{dr^2}{1 + \frac{r^2}{R^2}} \]

Having verified (6.6) let us now proceed to use it. We consider a stack of D7-branes embedded in an 8 dimensional space comprising \( \text{AdS}_5 \times \tilde{S}^3 \) where \( \tilde{S}^3 \subset S^5 \). The embedding is given by \( \phi = \text{constant} \) and \( \theta_3 \rightarrow \theta_3(u) \). The derivation of the DBI action was discussed in chapter 3. Recall that in the near-horizon limit, \( \alpha \rightarrow 0 \). In the absence of any background field, to lowest order in \( \alpha \):

\[ L_{\text{DBI}} \propto \sqrt{\text{det}(G_{ab})} \] (6.8)

where \( \text{det}(G_{ab}) \) denotes the induced metric on the brane. Since our metric is diagonal, the determinant is very easy to evaluate. Since \( \theta_3 \) is a function of \( u \),

\[ \frac{R^2}{u^2} du^2 + R^2 d\theta_3^2 = \frac{R^2}{u^2} du^2 + R^2 \theta_3'^2 du^2 \]

We multiply all of the individual terms in the metric and take the square root. To find the radial part of the lagrangian we just keep the terms that depend on \( u \) and absorb everything else including the constants in the proportionality sign. Consequently,

\[ \mathcal{L}_{\text{radial}} \propto \sqrt{\frac{u^2}{R^2} (1 + \frac{R^2}{4u^2})^2 u^6 (1 - \frac{R^2}{4u^2})^6 (\cos^2 \theta_3)^3 \frac{R^2}{u^2} (1 + u^2 \theta_3'^2)} \] (6.9)

\[ \implies \mathcal{L}_{\text{radial}} \propto (1 - \frac{R^4}{16u^4})(1 - \frac{R^2}{4u^2})^2 u^3 \cos^3 \theta_3 \sqrt{1 + u^2 \theta_3'^2} \] (6.10)

The AdS/CFT dictionary mentioned in [4] and [11] relates \( \theta_3 \) to the vacuum expectation value and the bare quark mass. The important relations are stated here:

\[ \sin \theta_3 = \frac{m}{u} + \frac{c_1}{u^3} - \frac{m \log u}{2u^2} + ... \] (6.11)

where \( c_1 \) is related to the vev as follows:

\[ \langle \overline{\Psi} \Psi \rangle \propto -2c_1 + m \log(\frac{m}{R}) \equiv -2c \] (6.12)

where \( c \) is the vev.

Next, we obtain the equation for \( \theta_3 \) using the Euler-Lagrange procedure. This gives us the following:

\[ 3u^3 \cos^2 \theta_3 \sin \theta_3 \sqrt{1 + u^2 \theta_3'^2} = \frac{d}{du} (u^3 \cos^3 \theta_3 \frac{u^2 \theta_3'}{\sqrt{1 + u^2 \theta_3'^2}}) \] (6.13)
The equation (6.10) is solved numerically\(^1\) in mathematica using a shooting technique as discussed in \([5]\). The solutions which represent the brane embeddings are plotted in figure 1.

There are two types of embeddings, the ones in red are called ball embeddings and the ones in black are called minkowski embeddings\(^2\). The embedding separating the two is called the critical embedding.

The asymptotic limits of these solutions were used to obtain \(c\) and \(m\) using (6.10). The details of this procedure are discussed in \([5]\). We will not concern ourselves with those details here. Defining \(\tilde{m} = \frac{m}{R}\) and \(\tilde{c} = \frac{c}{R}\), the plot obtained is given in figure 2.

The claim here is that the shift from minkowski embeddings to ball embeddings corresponds to a quantum phase transition. We will prove this in the next section.

\(^1\)In this thesis we have restricted ourselves to just deriving the equations given in this chapter, the solutions and the subsequent graphs were taken from \([4]\), \([5]\) and \([10]\).

\(^2\)The motivation for this naming may be found in \([4]\).
For now, it is clear from figure 2 that this transition is related to $\tilde{m}$. An alternate expression for $\tilde{m}$ is given in [4]:

$$\tilde{m} = \frac{\pi m_q R_3}{\sqrt{2}\lambda} \quad (6.14)$$

where $m_q$ is the bare quark mass, $\lambda$ is the gauge theory coupling constant and $R_3$ is the radius of $S^3 \subset AdS_5$. Small values of $\tilde{m}$, for a fixed bare quark mass, correspond to a small radius of $S^3$ and thence large casimir energy which causes the dissociation of bound quarks, as discussed in the previous chapter. Large values correspond to large radius and therefore smaller casimir energy as compared to the bare quark mass, thence favoring a bound quark state. Hence, at a critical value of $\tilde{m}$ we observe a confinement/deconfinement phase transition.

### 6.2 The phase transition

We will show that the shift between minkowski and ball embeddings corresponds to a quantum phase transitions and calculate the order of this transition using the appropriate critical exponents.

Let us first expand the metric (6.6) near the tip of the critical embedding: the dashed
curve in figure 1. For that, we do the following change in variables:

\[ u = \frac{1}{2}(1 + z); \quad y = \frac{\pi}{2} - \theta_3 \]  

(6.15)

\[ ds^2 = -\frac{1}{4R^2}(1 + z^2)(\frac{(1+z^2+R^2)^2}{(1+z)^2})d\tau^2 + \frac{1}{4}(1 + z)^2(\frac{((1+z)^2-R^2)^2}{(1+z)^2})d\Omega_3^{(1)^2} + \frac{R^2}{(1+z)^2}dz^2 + R^2dy^2 \ldots \]

\[ + \ldots \cos^2(\frac{\pi}{2})d\Omega_3^{(2)^2} + \sin^2(\frac{\pi}{2} - y)d\phi_3^2 \]

Since we are interested in looking at the immediate vicinity of the critical embedding, we expand the metric to leading order in y and z. For simplicity, we write R in suitable units such that the magnitude of R=1.

We end up with the following form for the metric:

\[ ds^2_{zoom} = -d\tau^2 + z^2d\Omega_3^{(1)^2} + dz^2 + dy^2 + y^2d\Omega_3^{(2)^2} + d\phi_3^2 \]  

(6.16)

where in each individual term, we have kept the term with the lowest power in z and y and additionally have used the fact that \( \sin(y) \approx y \) for small y. Further, to find the induced metric for the D7-brane, we take the same ansatz as before. Earlier, we took \( \theta_3 \to \theta_3(u) \) which now translates into \( y \to y(z) \). \( \phi \) is still constant so that gets rid of the \( d\phi_3^2 \) term in the induced metric for the brane. Since y is a function of z, the term \( dy^2 + dz^2 \to dz^2 + (1 + y^2)^2dy^2 \). The metric is still diagonal, hence the DBI lagrangian to leading order in \( \alpha' \) is obtained by multiplying the individual terms in the metric, multiplying the result by -1 and then taking the square root. This gives us:

\[ L_{zoom} = z^3y^3\sqrt{1 + y'^2} \]

(6.17)

This lagrangian is a special form the lagrangian:

\[ z^k y^n \sqrt{1 + y'^2} \]  

(6.18)

with \( k = 6 \) and \( n = 3 \). Let us derive the equations of motion of (6.17)

\[ \frac{\partial L}{\partial y} = \frac{\partial}{\partial z} \left( \frac{\partial L}{\partial y'} \right); \quad y' = \frac{\partial y}{\partial z} \]
\[ \Rightarrow n z^k y^{n-1} \sqrt{1 + y^2} = \frac{\partial}{\partial z} \left( \frac{z^k y^n y'}{2\sqrt{1+y^2}} \right) \]

\[ \Rightarrow n z^k y^{n-1} \sqrt{1 + y^2} = \frac{k z^{k-1} y y'}{2\sqrt{1+y^2}} + z^k n z^{n-1} y^2 + \frac{z^k y^n y'}{\sqrt{1+y^2}} + \frac{z^k y^n y'^2 y''}{\sqrt{1+y^2(1+y^2)}} \]

Dividing by \( z^{k-1} y^{n-1} \) throughout:

\[ n z \sqrt{1 + y^2} = \frac{k y y'}{2\sqrt{1+y^2}} + \frac{n z y^2}{\sqrt{1+y^2}} + \frac{z y y''}{\sqrt{1+y^2}} - \frac{z y y''}{(1+y^2)^{3/2}} \]

Multiplying by \((1 + y^2)^{1/2}\) and simplifying:

\[ n z = \frac{k}{2} y y' + z y y'' - \frac{z y y''}{(1+y^2)^{3/2}} \]

Now assume the following ansatz for \( y \):

\[ y = y (z) + \epsilon(z) = \sqrt{\frac{2n}{k}} z + \epsilon(z) \quad (6.19) \]

where \( \epsilon << 1 \)

\[ y' = \sqrt{\frac{2n}{k}} + \epsilon'; y'' = \epsilon''; y'^2 = \frac{2n}{k} + 2 \sqrt{\frac{2n}{k}} \epsilon' + \ldots \]

Inserting these in and using the small \( \epsilon \) approximation:

\[ n z \approx \frac{k}{2} \left( \sqrt{\frac{2n}{k}} z + \epsilon \right) \left( \sqrt{\frac{2n}{k}} + \epsilon' \right) + 2 \left( \sqrt{\frac{2n}{k}} z + \epsilon \right) \epsilon'' - z \left( 1 + \frac{2n}{k} + 2 \sqrt{\frac{2n}{k}} \epsilon' \right)^{-1} \left( \sqrt{\frac{2n}{k}} z + \epsilon \right) \ldots \]

\[ \ldots \left( \frac{2n}{k} + 2 \sqrt{\frac{2n}{k}} \epsilon' \right) \]

\[ \Rightarrow n z = \frac{k}{2} \left( \sqrt{\frac{2n}{k}} + \frac{2n}{k} z \epsilon + \sqrt{\frac{2n}{k}} + \frac{2n}{k} z^2 \epsilon'' - z \left( 1 + \frac{2n}{k} + 2 \sqrt{\frac{2n}{k}} \epsilon' \right)^{-1} \left( \frac{2n}{k} \sqrt{\frac{2n}{k}} \epsilon'' \right) \right) \]

\[ \Rightarrow n z = n z + \sqrt{\frac{2n}{k}} \left( \frac{k}{2} \right) z \epsilon' + \sqrt{\frac{2n}{k}} \left( \frac{k}{2} \right) \epsilon + \sqrt{\frac{2n}{k}} z^2 \epsilon'' - z^2 \left( 1 + \frac{2n}{k} + 2 \sqrt{\frac{2n}{k}} \epsilon' \right)^{-1} \left( \frac{2n}{k} \sqrt{\frac{2n}{k}} \epsilon'' \right) \]

Mutliplying by \((1 + \frac{2n}{k}) \sqrt{\frac{k}{2n}}\) throughout:

\[ 0 = \left( \frac{k}{2} z \epsilon' + \frac{k}{2} \epsilon + z^2 \epsilon'' \right) (1 + \frac{2n}{k}) - z^2 \left( \frac{2n}{k} \right) \epsilon'' \]

\[ \Rightarrow 0 = \frac{k}{2} z \epsilon' + \frac{k}{2} \epsilon + z^2 \epsilon'' + n z \epsilon' n \epsilon \]

\[ \Rightarrow 0 = z^2 \epsilon'' + (n + \frac{k}{2}) (z \epsilon' + \epsilon) \]

Hence, the equation of motion is given by:

\[ 0 = z^2 \epsilon'' (z) + (n + \frac{k}{2}) (z \epsilon' (z) + \epsilon (z)) \quad (6.20) \]
Note that if $\epsilon(z)$ satisfies the above equation then so does $\frac{1}{\mu}\epsilon(\mu z)$. The solution thus has a scaling property.

This equation is in the Cauchy-Euler form. To solve it consider the following: define $z = \exp(t)$, then $z^2 \frac{\partial^2}{\partial z^2} = \frac{\partial}{\partial t}(\frac{\partial}{\partial t} - 1)$ and $z \frac{\partial}{\partial z} = \frac{\partial}{\partial t}$

Since the equations are in one variable only, we can change all the partial derivatives to ordinary derivatives. As a result, our equation now becomes

$$(D(D-1) + (n + \frac{k}{2})D + (n + \frac{k}{2}))\epsilon = 0$$

where $D = \frac{d}{dt}$

The solution of such an equation we know is a linear combination of exponentials. It is given by:

$$\epsilon(z) = \text{const}_1 \exp(\alpha_1 t) + \text{const}_2 \exp(\alpha_2 t) \quad (6.21)$$

where $\alpha_{1/2}$ is given by the solution of the quadratic equation in $D$ given above:

$$\alpha_{1/2} = \frac{1}{2}[-(\frac{k}{2} + n - 1) \pm \sqrt{(\frac{k}{2} + n - 1)^2 - 4(\frac{k}{2} + n)}] \quad (6.22)$$

Replacing $t$ with $\ln z$, the solution becomes:

$$\epsilon(z) = \text{const}_1 z^{\alpha_1} + \text{const}_2 z^{\alpha_2} \quad (6.23)$$

Having solved the generic equation of motion, we return to our specific case where $k = 6$ and $n = 3$. In that case, from (6.22)

$$\alpha_1 = -2, \quad \alpha_2 = -3$$

Under the scaling condition mentioned above, the constants scale as follows:

$$\text{const}_1' = \text{const}_1 \mu^{\alpha_1 - 1}, \quad \text{const}_2' = \text{const}_2 \mu^{\alpha_2 - 1}$$

We know from our plot of figure 2 that the embedding of the brane given by $y^*$ corresponds to some bare quark mass $m^*$ and some condensate $c^*$. Consider now the following ansatz where we have assumed that $m - m^*$ and $c - c^*$ can be expanded analytically in terms of $\text{const}_1$ and $\text{const}_2$. The assumption is justified as long as we are looking at finite values of $m$ and $c$. Since we are looking at the vicinity of the critical embedding for which it is clear from figure 2 that these values are finite, this assumption is justified.
6.2. THE PHASE TRANSITION

\[ m - m^* = A_1 \text{const}_1 + A_2 \text{const}_2 + ... \]

\[ c - c^* = B_1 \text{const}_1 + B_2 \text{const}_2 + ... \]

for some constants \( A_1, A_2, B_1 \) and \( B_2 \).

Under the scaling transformation, this becomes:

\[ m' - m^* = A_1 \text{const}_1 \mu^{\alpha_1 - 1} + A_2 \text{const}_2 \mu^{\alpha_2 - 1} + ... \]

\[ c' - c^* = B_1 \text{const}_1 \mu^{\alpha_1 - 1} + B_2 \text{const}_2 \mu^{\alpha_2 - 1} + ... \]

From the equation in \( m' \) given above, we get that

\[ \mu \propto (m' - m^*)^{\frac{1}{\alpha_1 - 1}} \]

Inserting this in the equation for \( c' \), we get the following result:

\[ c - c^* = E_1 (m - m^*) + E_2 (m - m^*)^{\frac{\alpha_2 - 1}{\alpha_1 - 1}} \]  \hspace{1cm} (6.24)

where we have omitted the primed notation since it was there only to differentiate the various values of \( m \) and \( c \) in the first place and we do not need to do that anymore.

We have also defined a new set of coordinates \( E_1 \) and \( E_2 \). For the case when \( \alpha_1 = -2 \) and \( \alpha_2 = -3 \), this becomes:

\[ c - c^* = E_1 (m - m^*) + E_2 (m - m^*)^{\frac{1}{2}} \]  \hspace{1cm} (6.25)

It is clear from (6.25) that the second derivative of \( c - c^* \) is divergent. It has been discussed in [5] how the value of the fundamental condensate is related to free energy: it is proportional to the first derivative of the free energy energy. Therefore, in the relevant case, in the vicinity of the critical embedding, the third derivative of the free energy of the corresponding gauge theory becomes divergent. We conclude therefore that in moving across the critical embedding, from minkowski to ball embeddings, a phase transition takes place and that it is a third order transition. This, as discussed in the previous section, is a confinement/deconfinement quantum phase transition.

We will now proceed to look at this transition in the presence of an external magnetic field.
6.3 Phase Transition with an external magnetic field

Our metric contains a part $d\Omega_3^{(1)2}$ given by (6.4). For convenience, we wrote this as $d\Omega_3^{(1)2} = e^{(1)2} + e^{(2)2} + e^{(3)2}$. Now we apply the following background field $B = H e^{(1)} \wedge e^{(2)}$, i.e., the in 1 and 2 directions of $AdS_5$. We now have a non-zero Kalb-Ramond field to account for. The DBI lagrangian, to lowest order in $\alpha'$ is now given by:

$$\mathcal{L} \propto \sqrt{-\det(G_{ab} + B_{ab})}$$

(6.26)

The matrix is no longer diagonal but the determinant is still very straightforward to evaluate. The radial part of the lagrangian is calculated just like in (6.8-6.9). It is given by:

$$\mathcal{L}_{\text{radial}} \propto u \cos^3 \theta_3 (1 - \frac{R^4}{16u^4}) \sqrt{u^4(1 - \frac{R^2}{4u^2})^4 + H^2R^2 \sqrt{1 + u^2\theta_3'^2}}$$

(6.27)

As in the previous section, we look at the embedding in the vicinity of the critical embedding. We again define:

$$u = \frac{z}{2}(1 + z); y = \frac{\pi}{2} - \theta_3$$

(6.28)

and expand the metric to leading order in $z$ and $y$. Next we evaluate (6.26) which leads to the following expression:

$$\mathcal{L} \propto H z y^3 \sqrt{1 + y'^2}$$

(6.29)

We derived and solved the equations of motion for such lagrangians in the previous section. Note here that $k = 2$ and $n = 3$ and therefore from (6.22):

$$\alpha_{1/2} = -\frac{3}{2} \pm \frac{\sqrt{7}}{2}$$

(6.30)

Using these in (6.24), we see that since $\frac{\alpha_2 - 1}{\alpha_1 - 1}$ is complex, we would expect $c$ to be multivalued in some region. Indeed it is shown in [4] that this leads to a multivalued graph of $c$ against $m$ which has a spiral structure to it. Since small increments in $m$ cause $c$ to jump between the arms of this spiral graph, $c$ is discontinuous. Since the first derivative of the free energy is discontinuous, this is now a first order phase
6.4 Equation of state and phase diagram

From (6.26), using the euler-lagrange method, we obtain the equation of motion for $\theta_3$, just as in section 6.1. The resulting equation of motion, just as done earlier, is solved in mathematica and the asymptotes used to obtain values of $\tilde{c}$ and $\tilde{m}$, using the results from the AdS/CFT dictionary, given in (6.10) and (6.11). The results are given in the graphs for various values of H given in figure 3.

The graphs of $H = 0.4$ and $H = 1.0$ show a discontinuity in the value of $\tilde{c}$ for a certain value of $\tilde{m}$. Based on the arguments presented earlier, we conclude that this value corresponds to the phase transition. For the remaining two graphs, we use Maxwell’s equal area law [4] to determine the point of phase transition. The dotted vertical lines in the graph above indicate the value of $\tilde{m}$ for which the transition occurs.

Figure 3: Plot of $\tilde{c}$ versus $\tilde{m}$ for various values of $H$
We see that for $H = H_{cr} \approx 3.78$, the critical mass vanishes. Following on our discussion of chiral symmetry breaking in the previous chapter, we conclude that chiral symmetry breaking. The plot for $H$ against $\tilde{m}$—the critical mass is given in figure 4. Here we see that the critical mass decreases for increasing values of $H$. At $H = H_{cr}$, the critical mass is 0 after which the phase transition ceases to exist and the theory remains in a confined phase. The effect of the magnetic field overrides the dissociating effect of casimir energy from here on.

6.5 Meson spectra in an external magnetic field

We studied earlier how in the dual gravity picture, quarks are open strings with their endpoints lying on D-branes. One would expect, therefore, that fluctuations in the D-brane embeddings have some bearing on the quark states. It turns out that this is exactly what happens and that fluctuations in the brane embeddings can be studied in order to study the meson spectra of the theory.

6.5.1 Derivation of the fluctuation equations

We will study fluctuations about the embeddings discussed in section 6.1 which minimize the DBI action. Since we are expanding about the minima, the first order corrections vanish. Therefore, restricting ourselves to the leading order corrections, we look at the quadratic fluctuations of the D7-brane embedding. To do this, we begin by expanding the transverse coordinates as follows:

$$L(\rho) = L_0(\rho) + 2\pi \alpha' \chi(\rho); \phi_3(\rho) = 0 + 2\pi \alpha' \Phi(\rho)$$

where $L_0(\rho)$ corresponds to the embedding that is being expanded about. Since $\phi$ was a constant in our original choice of embedding, it is convenient to set it to 0. Recall that in deriving the induced metric, we also imposed the following condition: $\theta_3 \rightarrow \theta_3(u)$. Here, we have made the following change of variables:

$$\rho = u \cos \theta_3; L = u \sin \theta_3$$

Therefore, $u^2 = \rho^2 + L^2$.

In our earlier choice of coordinates, $u$ was the independent variable and $\theta_3$ was a function of $u$. For consistency, here too only one of $\rho$ and $L$ can be independent. We
choose \( \rho \) to be independent and \( L \rightarrow L(\rho) \) Furthermore, \( \phi_3 \) can also only be expanded in terms of a field which is a function of \( \rho \). To derive the form of the induced metric in terms of \( \rho \) and \( L \) we proceed as follows. In equation (6.6) everywhere where there is \( u^2 \) in the first two terms, we replace it with \( \rho^2 + L^2 \). The only term where we need to be a little more careful is the last term. Consider the following:

\[
\begin{align*}
    u^2 = \rho^2 + L^2 & \implies u du = \rho d\rho + (L'(\rho))d\rho \\
    & \implies du = (\cos \theta_3 + \sin \theta_3 L'(\rho))d\rho \\
    & \implies du^2 = d\rho^2(\cos^2 \theta_3 + \sin^2 \theta_3 L'(\rho)^2 + 2 \cos \theta_3 \sin \theta_3 L'(\rho))
\end{align*}
\]

From (6.31) we have:

\[
\cos \theta_3 = \frac{\rho}{u}
\]

Differentiating on both sides:

\[
-\sin \theta_3 d\theta_3 = \frac{d\rho}{u} - \frac{\rho}{u^2} du
\]

\[
\implies -\sin \theta_3 d\theta_3 = \frac{d\rho}{u} - \frac{\cos \theta_3}{u}(\cos \theta_3 d\rho + \sin \theta_3 L'(\rho)d\rho)
\]

\[
\implies u^2 d\theta_3^2 = \frac{d\rho^2}{\sin^2 \theta_3}(1 + \cos^4 \theta_3 + \cos^2 \theta_3^2 \sin^2 \theta_3 L'(\rho)^2 - 2 \cos^2 \theta_3 - 2 \cos \theta_3 \sin \theta_3 L'(\rho))
\]

From equation (6.3):

\[
\begin{align*}
    u^2 d\Omega_5^2 &= u^2 d\theta_3^2 + (u \cos \theta_3)^2 d\Omega_3^{(2)2} \\
    & \implies u^2 d\Omega_5^2 = u^2 d\theta_3^2 + \rho^2 d\Omega_3^{(2)2}
\end{align*}
\]

Let us now look at \( u^2 d\theta_3^2 + du^2 \):

\[
\implies \frac{d\rho^2}{\sin^2 \theta_3} + \frac{\cos^4 \theta_3}{\sin^2 \theta_3} \rho^2 + (\cos^2 \theta_3 + \sin^2 \theta_3 L'(\rho)^2) d\rho^2 + 2 \cos \theta_3 \sin \theta_3 L'(\rho) d\rho^2 + \frac{2 \cos^2 \theta_3}{\sin^2 \theta_3} L'(\rho) d\rho^2
\]

Now using \( 1 = \cos^2 \theta_3 + \sin^2 \theta_3 \) and simplifying, we get:

\[
\begin{align*}
    u^2 d\theta_3^2 + du^2 &= d\rho^2(1 + L'(\rho)^2)
\end{align*}
\]

Putting all of this analysis together, we can now finally write the full form of the induced metric in terms of \((L, \rho)\) coordinates:

\[
\begin{align*}
    ds_D^2 &= -\frac{\rho^2 + L^2}{R^2}(1 + \frac{R^2}{4(\rho^2 + L^2)}) d\tau^2 + (\rho^2 + L^2)(1 - \frac{R^2}{4(\rho^2 + L^2)})^2 d\Omega_3^{(1)2} + \frac{R^2}{\rho^2 + L^2} d\rho^2(1 + L'(\rho)^2) + ...
    \end{align*}
\]

(6.33)
We now proceed to derive the equations of motion of the brane. Remember that earlier, we were looking only at terms to zeroth order in \( \alpha' \) whereas now, since we are looking at quadratic fluctuations, we must take into account terms up to quadratic order in \( \alpha' \). In our discussion on D-branes, we described how D-branes source ramond-ramond charges and consequently a Dp-brane couples to a \( p+1 \) form background field known as the ramond-ramond field. It turns out that these terms become relevant at order \( \alpha' \) and \( \alpha'^2 \) and hence now they must be taken into account as well when writing the full brane action. The term in the action describing the ramond-ramond coupling is called the Wess-Zumino term. The derivation of this term in given in [8]. We skip the proof here and just state that the total lagrangian is given by:

\[
S = S_{DBI} + S_{WZ}
\]

\[
S_{DBI} \propto \int_{M^8} \left[ -\det(E_{ab} + 2\pi \alpha' F_{ab}) \right]^{\frac{1}{2}}
\]

\[
S_{WZ} \propto \int_{M^8} \left( \alpha'^2 (F_{(2)} \wedge F_{(2)} \wedge C_{(4)}) + \alpha' (F_{(2)} \wedge B_{(2)} \wedge P[C_{(4)}]) \right)
\]

where \( C_{(4)} \) is the four form ramond-ramond potential sourced by a stack of D3-branes. The expression for \( C_{(4)} \) is obtained by solving the relevant supergravity equations. It is given in [10]:

\[
g_s C_{(4)} = \frac{u^4}{R^4} d\tau \wedge dx^1 \wedge dx^2 \wedge dx^3
\]  

It is possible to show [10] that the pull back of its magnetic dual is given by:

\[
P[C_{(4)}] = \frac{-2\pi \alpha' \sin 2\psi}{g_s} \frac{R^4 L_0^2 (2\rho^2 + L_0^2)}{(\rho^2 + L_0^2)^2} \partial_\rho \Phi d\psi \wedge d\gamma \wedge d\beta \wedge d\rho
\]  

Since B has legs along directions 1 and 2, only the \( F_{03} \) part of \( F_{(2)} \) gives a non zero term in the second term of \( L_{WZ} \). Therefore, only the 03 (or 30) component of the world volume gauge field couples to \( \Phi \). Since here we are only interested in terms that couple to \( \Phi \), we can set all other tems of \( F_{(2)} \) to 0. It then follows that the first term in \( L_{WZ} \) is identically zero for our consideration. As a result, the Wess-Zumino part contributes only one term to our lagrangian which is as follows:

\[
\mathcal{L}_{\Phi A} \propto -H(\partial_\rho K(\rho))\Phi F_{03}
\]
where we have integrated over $\rho$ by parts and have defined:

$$K(\rho) = \frac{R^4 L_0^2 (2\rho^2 + L_0^2)}{(\rho^2 + L_0^2)^2}$$

(6.37)

Before deriving the lagrangian terms from $L_{DBI}$, we simplify it a little more. We plug the ansatz given in (6.30) into the $E_{ab} = g_{ab} + B_{ab}$ term in the lagrangian and collect the terms with the same power of $\alpha'$. It is a simple but length exercise to show that it reduces to the following:

$$E_{ab} = E_{ab}^0 + 2\pi \alpha' E_{ab}^1 + (2\pi \alpha')^2 E_{ab}^2$$

(6.38)

where

$$E_{ab}^0 = g_{ab}(L_0(\rho)) + B_{ab}$$

$$E_{ab}^1 = \frac{R^2}{\rho^2 + L_0^2} L_0'(\rho)(\partial_a \chi \delta_b^\rho + \partial_b \chi \delta_a^\rho) + (\partial L_0 g_{ab})\chi$$

and

$$E_{ab}^2 = \frac{R^2}{\rho^2 + L_0^2}(\partial_a \chi \partial_b \chi + L_0^2 \partial_a \Phi \partial_b \Phi) + (\partial L_0 (-\frac{R^2}{\rho^2 + L_0^2}) L_0') (\partial_a \chi \delta_b^\rho + \partial_b \chi \delta_a^\rho) \chi + \frac{1}{2} \partial_L g_{ab} \chi^2$$

where $g_{ab}$ is the induced metric on the brane volume without the ansatz having been plugged in. $B_{ab}$ is the Kalb-Ramond field on the D7-brane world volume.

For later convenience, let us split the inverse matrix of $E_{ab}^0$ as follows:

$$(E_{ab}^0)^{-1} = S_{ab} + J_{ab}$$

where $S$ is a symmetric matrix given by:

$$S_{ab} = \text{diag}[g_{tt}^{-1}, g_{33}^{-1}, \frac{g_{33}}{\rho^2 + H^2}, g_{33}^{-1}, g_{\rho\rho}^{-1}, g_{\phi_2\phi_2}^{-1}, g_{\psi_2\psi_2}^{-1}]$$

where

$$g_{tt} = \frac{\rho^2 + L_0^2}{R^2}(1 + \frac{R^2}{\rho^2 + L_0^2})^2, g_{33} = \frac{\rho^2 + L_0^2}{R^2}(1 - \frac{R^2}{\rho^2 + L_0^2})^2, g_{\rho\rho} = \frac{R^2}{\rho^2 + L_0^2}(1 + L_0'(\rho)^2)$$

The rest of the terms can be read off from (6.32) but they will not concern us here directly.

and $J$ is an antisymmetric matrix given by:

$$J_{ab} = \frac{H^2}{g_{33} + H^2} (\delta_a^\phi \delta_b^\psi - \delta_a^\psi \delta_b^\phi)$$
One can easily verify that this matrix is indeed the inverse of \( E^0 \) by multiplying it with it and obtaining the identity. Having done the groundwork, we now proceed to derive the explicit form of \( L_{DBI} \).

Calculating \( L_{DBI} \) involves evaluating a determinant. To do this, we use a well known expansion for the determinant to write it as follows:

\[
L^{(2)}_{DBI} = \frac{1}{2} \sqrt{-det E^0} \left[ Tr(E^{(0)-1} E^{(2)}) + \frac{1}{4} (Tr(E^{(0)-1} E^{(1)}))^2 + \frac{1}{4} (Tr(E^{(0)-1} F))^2 + \frac{1}{2} (E^{(0)-1} E^{(1)}) Tr(E^{(0)-1} F) \right]
\]

\((6.39)\)

\[
...-\frac{1}{2} Tr(E^{(0)-1} E^{(1)})^2 - \frac{1}{2} Tr(E^{(0)-1} F)^2 - Tr(E^{(0)-1} E^{(1)} E^{(0)-1} F) \]

It is now easy to show that the lagrangian reduces to the following terms:

\[
L_{\chi \chi} \propto \frac{1}{2} g(\rho) G_{LL} S^{tt} \partial_t \chi \partial_t \chi + \frac{1}{2} g(\rho) G_{LL} S^{\rho \rho} \partial_{\rho} \partial_{\rho} + \frac{1}{2} g(\rho) [\partial L_0 (\partial L_0 \log g(\rho))] \]

\[(6.40)\]

\[
L_{\Phi \Phi} \propto \frac{1}{2} g(\rho) g_{\phi_3 \phi_3} (S^{tt} \partial_t \Phi \partial_t \Phi + S^{\rho \rho} \partial_{\rho} \Phi \partial_{\rho} \Phi) \]

\[(6.41)\]

and

\[
L_{A \Phi} = \frac{1}{4} g(\rho) S^{tt} S^{33} (F_{3t} F_{3t} + F_{3\rho} F_{3\rho}) \]

\[(6.42)\]

where

\[
G_{\phi_3 \phi_3} = \frac{R^2 L_0^2}{\rho^2 + L_0^2}; \quad g(\rho) = \sqrt{-det E^0}; \quad G_{LL} = \frac{R^2}{\rho^2 + L_0^2}
\]

It is possible to show that there is a gauge in which modes which propagate only in time and \( \rho \) couple to \( \Phi \) only through the \( A_3 \) component of \( F \) and hence to get the spectrum of \( \Phi \) you can set the rest of the components to zero. We now have our full lagrangian with the additional knowledge that the only known zero component of the world volume gauge field is \( A_3 \) and therefore \( F_{30} = -\partial_0 A_3 \). Recall here that the fields are only explicit functions of \( t \) and \( \rho \). Let us now proceed to derive the equations of motion. Consider first the equation corresponding to (6.42) and (6.36). From the euler-lagrange procedure, we get:
Let us now move to equation (6.40). The corresponding Euler-Lagrange equation is:

\[ \partial_{t}(g(\rho)\frac{R^{2}}{\rho^{2}+L_{0}^{2}}(1-\frac{R^{2}}{4(\rho^{2}+L_{0}^{2}))^{2}})\rho^{2}+\rho^{2}L_{0}^{2}(1+L_{0}^{2}(\rho))^{-1}\partial_{\rho}A_{3})+... \]  

(6.43)

\[ \partial_{t}(g(\rho)\frac{R^{2}}{\rho^{2}+L_{0}^{2}}(1+\frac{R^{2}}{4(\rho^{2}+L_{0}^{2}))^{2}})(1-\frac{R^{2}}{4(\rho^{2}+L_{0}^{2}))^{2}})\rho^{2}\partial_{t}A_{3})+\partial_{t}(-H(\partial_{\rho}K(\rho))\Phi) = 0 \]

Now differentiate both sides with respect to \(\partial_{t}\), dividing by \(g(\rho)\) throughout and using the ansatz \(\partial_{t}^{2}\propto w^{2}\) we get:

\[ \frac{1}{g(\rho)}\partial_{\rho}(\frac{g(\rho)}{1+L_{0}^{2}}\frac{\partial_{\rho}F_{03}}{(\rho^{2}+L_{0}^{2})^{2}}) + \frac{R^{4}}{(\rho^{2}+L_{0}^{2})^{2}(1-\frac{R^{4}}{16(\rho^{2}+L_{0}^{2}))^{2}}w^{2}F_{03}... \]  

(6.44)

\[ ... - H\partial_{\rho}Kw^{2}\Phi \]

Let us now move to equation (6.40). The corresponding Euler-Lagrange equation is:

\[ g(\rho)(\frac{R^{2}}{\rho^{2}+L_{0}^{2}})(\frac{R^{2}}{\rho^{2}+L_{0}^{2}})(1+\frac{R^{2}}{4(\rho^{2}+L_{0}^{2}))^{2}})(1+L_{0}^{2})^{-1}\omega^{2}\chi + \partial_{\rho}[g(\rho)(\frac{R^{2}}{\rho^{2}+L_{0}^{2}})(1+L_{0}^{2})^{-2}\partial_{\rho}\chi] = ... \]  

(6.45)

\[ ... \frac{1}{g(\rho)}\partial_{L_{0}}[\partial_{L_{0}}\log g(\rho) - \frac{L_{0}}{1+L_{0}^{2}}\partial_{\rho}(\partial_{L_{0}}\log g(\rho))]\chi \]

Dividing throughout by \(g(\rho)\):

\[ \frac{1}{g(\rho)}\partial_{\rho}(\frac{g(\rho)\partial_{\chi}}{(1+L_{0}^{2})^{2}}) + \frac{R^{4}w^{2}\chi}{(1+L_{0}^{2})^{2}(\rho^{2}+L_{0}^{2})^{2}(1+\frac{R^{4}}{4(\rho^{2}+L_{0}^{2}))^{2}}... \]  

(6.46)

\[ ... - (\partial_{L_{0}}(\partial_{L_{0}}\log g(\rho)) - \frac{L_{0}}{1+L_{0}^{2}}\partial_{\rho}(\partial_{L_{0}}\log g(\rho))\chi = 0 \]

Finally, we consider \(\mathcal{L}_{\Phi\Phi}\) and \(\mathcal{L}_{\Phi A}\). Repeating the same process, we get the following:

\[ \frac{1}{g(\rho)}\partial_{\rho}(\frac{g(\rho)L_{0}^{2}\partial_{\rho}\Phi}{(1+L_{0}^{2})}) + \frac{L_{0}^{2}R^{4}w^{2}\Phi}{(\rho^{2}+L_{0}^{2})^{2}(1+\frac{R^{4}}{4(\rho^{2}+L_{0}^{2}))^{2}} = \frac{H\partial_{\rho}KF_{03}}{g(\rho)} \]  

(6.47)

This completes our derivation of the fluctuation equations of motion. These equations can be numerically solved for \(\chi\) and \(\phi\) as done in [4]. The normalizability-condition for fluctuation modes (coefficient of the first term in the asymptotic expansion at infinity goes to zero) allows us to extract the allowed values of \(\omega^{3}\) for various embeddings.

\(^{3}L_{0}\) was used just to denote the unperturbed embedding which is the same as \(L\) which was defined in (6.32).
Therefore, for each embedding, we can obtain a relation between $\omega$ (the energy) and $H$. This is used to study the meson spectra corresponding to fluctuations along $L$ and $\phi$. A detailed discussion of the spectra is given in [4]. It is demonstrated in [4] that for the $\Phi$ fluctuations at larger values of $H$ and small values of $m$ (the bare mass), the energy levels split up to give a Zeeman like effect. At sufficiently low values of $m$, the Zeeman splitting is strong and the energy levels intersect giving rise to what is called level crossing. The results are shown in figure 4.

The interested reader is referred to [4] for these details. The important thing to note here for our purposes is that an analysis based purely using the gravity part of the AdS/CFT correspondence enables us to study phenomena that are described by the dual gauge theory, for example, Zeeman splitting and level crossing. This demonstrates how the AdS/CFT correspondence can be applied to problems of strongly coupled gauge theories.
6.6 Conclusion

The strong-weak duality proposed by the AdS/CFT correspondence offers a tractable way to study phenomena that are described by strongly coupled gauge theories including quark confinement and chiral symmetry breaking. Using the correspondence, we were able to see how an analysis based on supergravity leads to insights about quark confinement/deconfinement transitions, zeeman splitting and level crossing in the dual gauge theory, thus demonstrating the power of the AdS/CFT conjecture.
CHAPTER 6. LARGE N FLAVORED GAUGE THEORY ON A COMPACT MANIFOLD WITH AN EXTERNAL MAGNETIC FIELD: THE APPLICATION.
Bibliography


